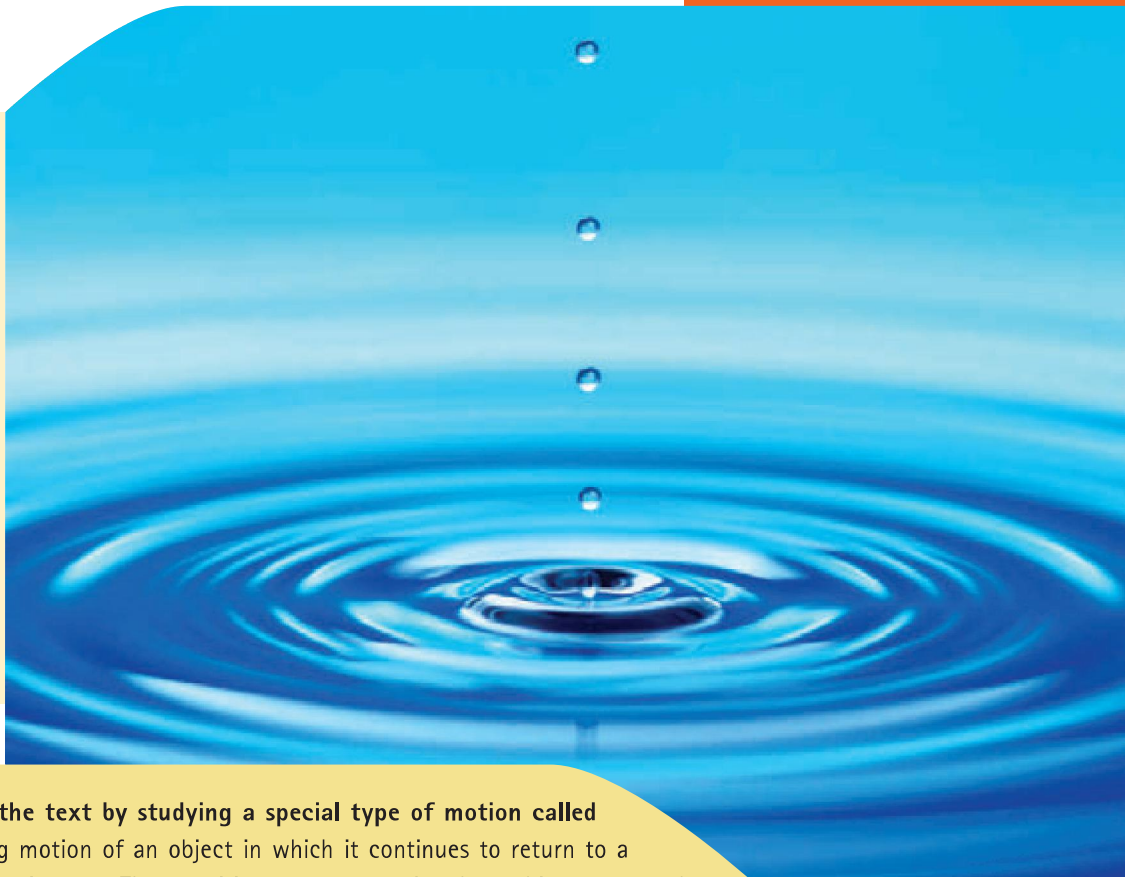# Oscillations and Mechanical Waves

Falling drops of water cause a water surface to oscillate. These oscillations are associated with circular waves moving away from the point at which the drops fall. In Part 2 of the text, we will explore the principles related to oscillations and waves. *(Marga Buschbell Steeger/Photographer's Choice/ Getty Images)*

We begin this new part of the text by studying a special type of motion called *periodic* motion, the repeating motion of an object in which it continues to return to a given position after a fixed time interval. The repetitive movements of such an object are called *oscillations.* We will focus our attention on a special case of periodic motion called *simple harmonic motion.* All periodic motions can be modeled as combinations of simple harmonic motions.

Simple harmonic motion also forms the basis for our understanding of *mechanical waves.* Sound waves, seismic waves, waves on stretched strings, and water waves are all produced by some source of oscillation. As a sound wave travels through the air, elements of the air oscillate back and forth; as a water wave travels across a pond, elements of the water oscillate up and down and backward and forward. The motion of the elements of the medium bears a strong resemblance to the periodic motion of an oscillating pendulum or an object attached to a spring.

To explain many other phenomena in nature, we must understand the concepts of oscillations and waves. For instance, although skyscrapers and bridges appear to be rigid, they actually oscillate, something the architects and engineers who design and build them must take into account. To understand how radio and television work, we must understand the origin and nature of electromagnetic waves and how they propagate through space. Finally, much of what scientists have learned about atomic structure has come from information carried by waves. Therefore, we must first study oscillations and waves if we are to understand the concepts and theories of atomic physics. ■

Oscillatory Motion

The London Millennium Bridge over the River Thames in London. On opening day of the bridge, pedestrians noticed a swinging motion of the bridge, leading to its being named the "Wobbly Bridge." The bridge was closed after two days and remained closed for two years. Over 50 *tuned mass dampers* were added to the bridge: the pairs of spring-loaded structures on top of the cross members (arrow). We will study both oscillations and damping of oscillations in this chapter. *(Monkey Business Images/ Shutterstock.com)*

Periodic motion is motion of an object that regularly returns to a given position after a fixed time interval. With a little thought, we can identify several types of periodic motion in everyday life. Your car returns to the driveway each afternoon. You return to the dinner table each night to eat. A bumped chandelier swings back and forth, returning to the same position at a regular rate. The Earth returns to the same position in its orbit around the Sun each year, resulting in the variation among the four seasons.

A special kind of periodic motion occurs in mechanical systems when the force acting on an object is proportional to the position of the object relative to some equilibrium position. If this force is always directed toward the equilibrium position, the motion is called *simple harmonic motion*, which is the primary focus of this chapter.

15.1 Motion of an Object Attached to a Spring

As a model for simple harmonic motion, consider a block of mass *m* attached to the end of a spring, with the block free to move on a frictionless, horizontal surface

(Fig. 15.1). When the spring is neither stretched nor compressed, the block is at rest at the position called the **equilibrium position** of the system, which we identify as $x = 0$ (Fig. 15.1b). We know from experience that such a system oscillates back and forth if disturbed from its equilibrium position.

We can understand the oscillating motion of the block in Figure 15.1 qualitatively by first recalling that when the block is displaced to a position x, the spring exerts on the block a force that is proportional to the position and given by **Hooke's law** (see Section 7.4):

$$F_s = -kx \qquad (15.1)$$

◀ Hooke's law

We call F_s a **restoring force** because it is always directed toward the equilibrium position and therefore *opposite* the displacement of the block from equilibrium. That is, when the block is displaced to the right of $x = 0$ in Figure 15.1a, the position is positive and the restoring force is directed to the left. When the block is displaced to the left of $x = 0$ as in Figure 15.1c, the position is negative and the restoring force is directed to the right.

When the block is displaced from the equilibrium point and released, it is a particle under a net force and consequently undergoes an acceleration. Applying the particle under a net force model to the motion of the block, with Equation 15.1 providing the net force in the x direction, we obtain

$$\sum F_x = ma_x \;\;\rightarrow\;\; -kx = ma_x$$

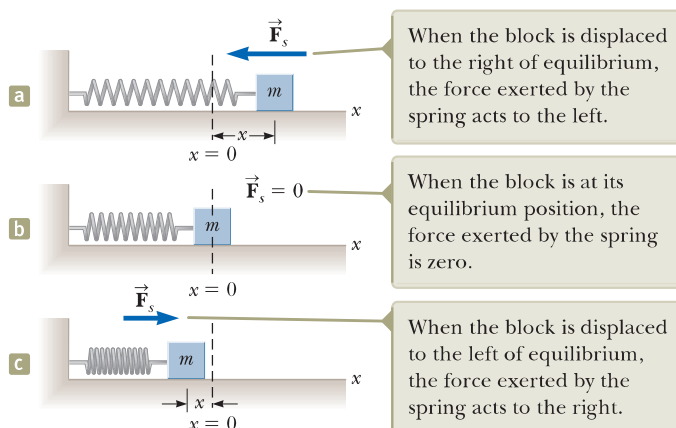$$a_x = -\frac{k}{m}x \qquad (15.2)$$

That is, the acceleration of the block is proportional to its position, and the direction of the acceleration is opposite the direction of the displacement of the block from equilibrium. Systems that behave in this way are said to exhibit **simple harmonic motion.** An object moves with simple harmonic motion whenever its acceleration is proportional to its position and is oppositely directed to the displacement from equilibrium.

If the block in Figure 15.1 is displaced to a position $x = A$ and released from rest, its *initial* acceleration is $-kA/m$. When the block passes through the equilibrium position $x = 0$, its acceleration is zero. At this instant, its speed is a maximum because the acceleration changes sign. The block then continues to travel to the left of equilibrium with a positive acceleration and finally reaches $x = -A$, at which time its acceleration is $+kA/m$ and its speed is again zero as discussed in Sections 7.4 and 7.9. The block completes a full cycle of its motion by returning to the original position, again passing through $x = 0$ with maximum speed. Therefore, the block oscillates between the turning points $x = \pm A$. In the absence of

Pitfall Prevention 15.1

The Orientation of the Spring Figure 15.1 shows a *horizontal* spring, with an attached block sliding on a frictionless surface. Another possibility is a block hanging from a *vertical* spring. All the results we discuss for the horizontal spring are the same for the vertical spring with one exception: when the block is placed on the vertical spring, its weight causes the spring to extend. If the resting position of the block is defined as $x = 0$, the results of this chapter also apply to this vertical system.

a When the block is displaced to the right of equilibrium, the force exerted by the spring acts to the left.

b When the block is at its equilibrium position, the force exerted by the spring is zero.

c When the block is displaced to the left of equilibrium, the force exerted by the spring acts to the right.

Figure 15.1 A block attached to a spring moving on a frictionless surface.

friction, this idealized motion will continue forever because the force exerted by the spring is conservative. Real systems are generally subject to friction, so they do not oscillate forever. We shall explore the details of the situation with friction in Section 15.6.

Quick Quiz 15.1   A block on the end of a spring is pulled to position $x = A$ and released from rest. In one full cycle of its motion, through what total distance does it travel? **(a)** $A/2$ **(b)** A **(c)** $2A$ **(d)** $4A$

15.2 Analysis Model: Particle in Simple Harmonic Motion

The motion described in the preceding section occurs so often that we identify the **particle in simple harmonic motion** model to represent such situations. To develop a mathematical representation for this model, we will generally choose x as the axis along which the oscillation occurs; hence, we will drop the subscript-x notation in this discussion. Recall that, by definition, $a = dv/dt = d^2x/dt^2$, so we can express Equation 15.2 as

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{15.3}$$

If we denote the ratio k/m with the symbol ω^2 (we choose ω^2 rather than ω so as to make the solution we develop below simpler in form), then

$$\omega^2 = \frac{k}{m} \tag{15.4}$$

and Equation 15.3 can be written in the form

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{15.5}$$

> **Pitfall Prevention 15.2**
>
> **A Nonconstant Acceleration** The acceleration of a particle in simple harmonic motion is not constant. Equation 15.3 shows that its acceleration varies with position x. Therefore, we *cannot* apply the kinematic equations of Chapter 2 in this situation.

Let's now find a mathematical solution to Equation 15.5, that is, a function $x(t)$ that satisfies this second-order differential equation and is a mathematical representation of the position of the particle as a function of time. We seek a function whose second derivative is the same as the original function with a negative sign and multiplied by ω^2. The trigonometric functions sine and cosine exhibit this behavior, so we can build a solution around one or both of them. The following cosine function is a solution to the differential equation:

◄ Position versus time for a particle in simple harmonic motion

$$x(t) = A \cos{(\omega t + \phi)} \tag{15.6}$$

where A, ω, and ϕ are constants. To show explicitly that this solution satisfies Equation 15.5, notice that

$$\frac{dx}{dt} = A\frac{d}{dt}\cos{(\omega t + \phi)} = -\omega A \sin{(\omega t + \phi)} \tag{15.7}$$

> **Pitfall Prevention 15.3**
>
> **Where's the Triangle?** Equation 15.6 includes a trigonometric function, a *mathematical function* that can be used whether it refers to a triangle or not. In this case, the cosine function happens to have the correct behavior for representing the position of a particle in simple harmonic motion.

$$\frac{d^2x}{dt^2} = -\omega A\frac{d}{dt}\sin{(\omega t + \phi)} = -\omega^2 A \cos{(\omega t + \phi)} \tag{15.8}$$

Comparing Equations 15.6 and 15.8, we see that $d^2x/dt^2 = -\omega^2 x$ and Equation 15.5 is satisfied.

The parameters A, ω, and ϕ are constants of the motion. To give physical significance to these constants, it is convenient to form a graphical representation of the motion by plotting x as a function of t as in Figure 15.2a. First, A, called the **amplitude** of the motion, is simply the maximum value of the position of the particle in

either the positive or negative x direction. The constant ω is called the **angular frequency,** and it has units[1] of radians per second. It is a measure of how rapidly the oscillations are occurring; the more oscillations per unit time, the higher the value of ω. From Equation 15.4, the angular frequency is

$$\omega = \sqrt{\frac{k}{m}}$$

(15.9)

The constant angle ϕ is called the **phase constant** (or initial phase angle) and, along with the amplitude A, is determined uniquely by the position and velocity of the particle at $t = 0$. If the particle is at its maximum position $x = A$ at $t = 0$, the phase constant is $\phi = 0$ and the graphical representation of the motion is as shown in Figure 15.2b. The quantity $(\omega t + \phi)$ is called the **phase** of the motion. Notice that the function $x(t)$ is periodic and its value is the same each time ωt increases by 2π radians.

Equations 15.1, 15.5, and 15.6 form the basis of the mathematical representation of the particle in simple harmonic motion model. If you are analyzing a situation and find that the force on an object modeled as a particle is of the mathematical form of Equation 15.1, you know the motion is that of a simple harmonic oscillator and the position of the particle is described by Equation 15.6. If you analyze a system and find that it is described by a differential equation of the form of Equation 15.5, the motion is that of a simple harmonic oscillator. If you analyze a situation and find that the position of a particle is described by Equation 15.6, you know the particle undergoes simple harmonic motion.

Figure 15.2 (a) An x–t graph for a particle undergoing simple harmonic motion. The amplitude of the motion is A, and the period (defined in Eq. 15.10) is T. (b) The x–t graph for the special case in which $x = A$ at $t = 0$ and hence $\phi = 0$.

Quick Quiz 15.2 Consider a graphical representation (Fig. 15.3) of simple harmonic motion as described mathematically in Equation 15.6. When the particle is at point Ⓐ on the graph, what can you say about its position and velocity? **(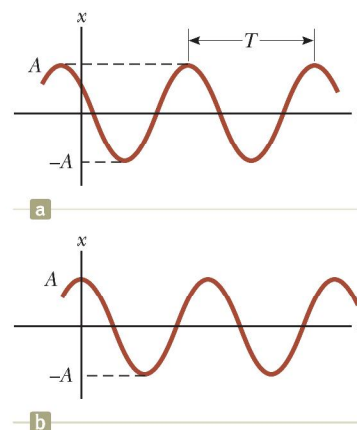a)** The position and velocity are both positive. **(b)** The position and velocity are both negative. **(c)** The position is positive, and the velocity is zero. **(d)** The position is negative, and the velocity is zero. **(e)** The position is positive, and the velocity is negative. **(f)** The position is negative, and the velocity is positive.

Figure 15.3 (Quick Quiz 15.2) An x–t graph for a particle undergoing simple harmonic motion. At a particular time, the particle's position is indicated by Ⓐ in the graph.

Quick Quiz 15.3 Figure 15.4 shows two curves representing particles undergoing simple harmonic motion. The correct description of these two motions is that the simple harmonic motion of particle B is **(a)** of larger angular frequency and larger amplitude than that of particle A, **(b)** of larger angular frequency and smaller amplitude than that of particle A, **(c)** of smaller angular frequency and larger amplitude than that of particle A, or **(d)** of smaller angular frequency and smaller amplitude than that of particle A.
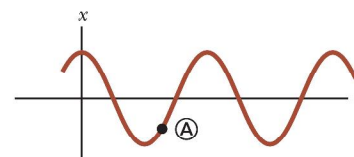
Let us investigate further the mathematical description of simple harmonic motion. The **period** T of the motion is the time interval required for the particle to go through one full cycle of its motion (Fig. 15.2a). That is, the values of x and v for the particle at time t equal the values of x and v at time $t + T$. Because the phase increases by 2π radians in a time interval of T,

$$[\omega(t + T) + \phi] - (\omega t + \phi) = 2\pi$$

Particle A

Particle B

Figure 15.4 (Quick Quiz 15.3) Two x–t graphs for particles undergoing simple harmonic motion. The amplitudes and frequencies are different for the two particles.
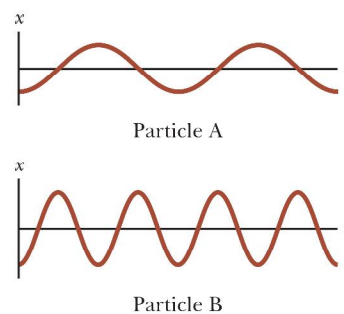
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[1]We have seen many examples in earlier chapters in which we evaluate a trigonometric function of an angle. The argument of a trigonometric function, such as sine or cosine, *must* be a pure number. The radian is a pure number because it is a ratio of lengths. Angles in degrees are pure numbers because the degree is an artificial "unit"; it is not related to measurements of lengths. The argument of the trigonometric function in Equation 15.6 must be a pure number. Therefore, ω *must* be expressed in radians per second (and not, for example, in revolutions per second) if t is expressed in seconds. Furthermore, other types of functions such as logarithms and exponential functions require arguments that are pure numbers.

Simplifying this expression gives $\omega T = 2\pi$, or

$$T = \frac{2\pi}{\omega} \tag{15.10}$$

The inverse of the period is called the **frequency** f of the motion. Whereas the period is the time interval per oscillation, the frequency represents the number of oscillations the particle undergoes per unit time interval:

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{15.11}$$

The units of f are cycles per second, or **hertz** (Hz). Rearranging Equation 15.11 gives

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{15.12}$$

Equations 15.9 through 15.11 can be used to express the period and frequency of the motion for the particle in simple harmonic motion in terms of the characteristics m and k of the system as

Period ▶
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{15.13}$$

Frequency ▶
$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{15.14}$$

That is, the period and frequency depend *only* on the mass of the particle and the force constant of the spring and *not* on the parameters of the motion, such as A or ϕ. As we might expect, the frequency is larger for a stiffer spring (larger value of k) and decreases with increasing mass of the particle.

We can obtain the velocity and acceleration[2] of a particle undergoing simple harmonic motion from Equations 15.7 and 15.8:

Velocity of a particle in ▶
simple harmonic motion
$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi) \tag{15.15}$$

Acceleration of a particle in ▶
simple harmonic motion
$$a = \frac{d^2 x}{dt^2} = -\omega^2 A \cos(\omega t + \phi) \tag{15.16}$$

From Equation 15.15, we see that because the sine and cosine functions oscillate between ± 1, the extreme values of the velocity v are $\pm \omega A$. Likewise, Equation 15.16 shows that the extreme values of the acceleration a are $\pm \omega^2 A$. Therefore, the *maximum* values of the magnitudes of the velocity and acceleration are

Maximum magnitudes of ▶
velocity and acceleration in
simple harmonic motion
$$v_{\max} = \omega A = \sqrt{\frac{k}{m}}\, A \tag{15.17}$$

$$a_{\max} = \omega^2 A = \frac{k}{m}\, A \tag{15.18}$$

Figure 15.5a plots position versus time for an arbitrary value of the phase constant. The associated velocity–time and acceleration–time curves are illustrated in Figures 15.5b and 15.5c, respectively. They show that the phase of the velocity differs from the phase of the position by $\pi/2$ rad, or 90°. That is, when x is a maximum or a minimum, the velocity is zero. Likewise, when x is zero, the speed is a maximum. Furthermore, notice that the phase of the acceleration differs from the phase of the position by π radians, or 180°. For example, when x is a maximum, a has a maximum magnitude in the opposite direction.

---

[2]Because the motion of a simple harmonic oscillator takes place in one dimension, we denote velocity as v and acceleration as a, with the direction indicated by a positive or negative sign as in Chapter 2.

Quick Quiz 15.4 An object of mass m is hung from a spring and set into oscilla-tion. The period of the oscillation is measured and recorded as T. The object of mass m is removed and replaced with an object of mass $2m$. When this object is set into oscillation, what is the period of the motion? **(a)** $2T$ **(b)** $\sqrt{2}\,T$ **(c)** T **(d)** $T/\sqrt{2}$ **(e)** $T/2$

Equation 15.6 describes simple harmonic motion of a particle in general. Let's now see how to evaluate the constants of the motion. The angular frequency ω is evaluated using Equation 15.9. The constants A and ϕ are evaluated from the ini-tial conditions, that is, the state of the oscillator at $t = 0$.

Suppose a block is set into motion by pulling it from equilibrium by a distance A and releasing it from rest at $t = 0$ as in Figure 15.6. We must then require our solu-tions for $x(t)$ and $v(t)$ (Eqs. 15.6 and 15.15) to obey the initial conditions that $x(0) = A$ and $v(0) = 0$:

$$x(0) = A \cos \phi = A$$

$$v(0) = -\omega A \sin \phi = 0$$

These conditions are met if $\phi = 0$, giving $x = A \cos \omega t$ as our solution. To check this solution, notice that it satisfies the condition that $x(0) = A$ because $\cos 0 = 1$.

The position, velocity, and acceleration of the block versus time are plotted in Figure 15.7a for this special case. The acceleration reaches extreme values of $\mp \omega^2 A$ when the position has extreme values of $\pm A$. Furthermore, the velocity has extreme values of $\pm \omega A$, which both occur at $x = 0$. Hence, the quantitative solution agrees with our qualitative description of this system.
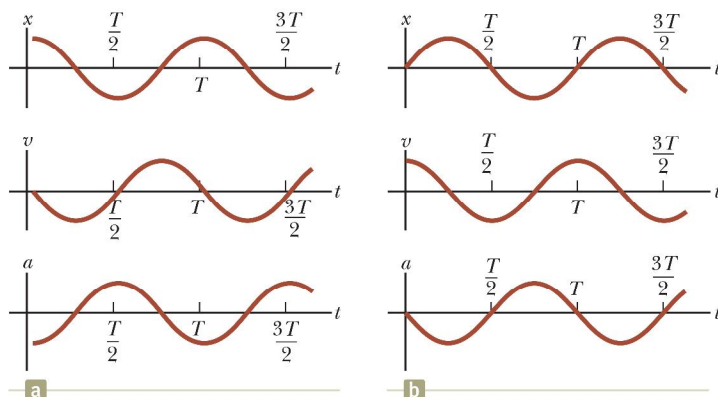
Let's consider another possibility. Suppose the system is oscillating and we define $t = 0$ as the instant the block passes through the unstretched position of the spring while moving to the right (Fig. 15.8). In this case, our solutions for $x(t)$ and $v(t)$ must obey the initial conditions that $x(0) = 0$ and $v(0) = v_i$:

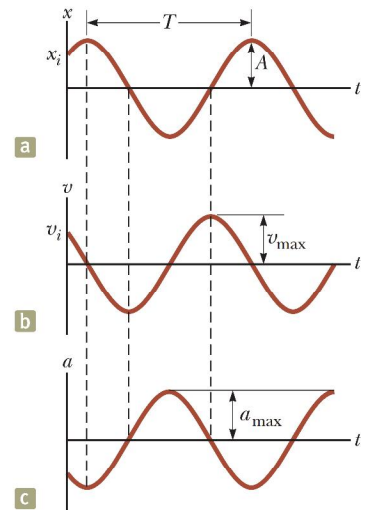$$x(0) = A \cos \phi = 0$$

$$v(0) = -\omega A \sin \phi = v_i$$

The first of these conditions tells us that $\phi = \pm \pi/2$. With these choices for ϕ, the second condition tells us that $A = \mp v_i/\omega$. Because the initial velocity is positive and the amplitude must be positive, we must have $\phi = -\pi/2$. Hence, the solution is

$$x = \frac{v_i}{\omega} \cos\left(\omega t - \frac{\pi}{2}\right)$$

The graphs of position, velocity, and acceleration versus time for this choice of $t = 0$ are shown in Figure 15.7b. Notice that these curves are the same as those in Figure

Figure 15.5 Graphical repre-sentation of simple harmonic motion. (a) Position versus time. (b) Velocity versus time. (c) Accel-eration versus time. Notice that at any specified time the velocity is 90° out of phase with the position and the acceleration is 180° out of phase with the position.
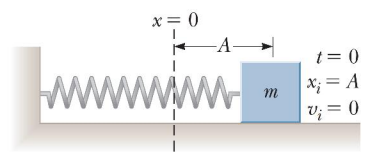
Figure 15.6 A block–spring system that begins its motion from rest with the block at $x = A$ at $t = 0$.
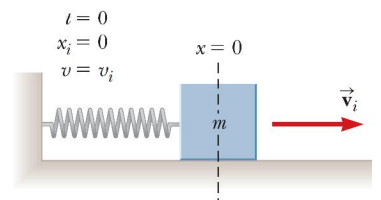
Figure 15.8 The block–spring system is undergoing oscillation, and $t = 0$ is defined at an instant when the block passes through the equilibrium position $x = 0$ and is moving to the right with speed v_i.

Figure 15.7 (a) Position, velocity, and acceleration versus time for the block in Figure 15.6 under the initial conditions that at $t = 0$, $x(0) = A$, and $v(0) = 0$. (b) Position, velocity, and acceleration ver-sus time for the block in Figure 15.8 under the initial conditions that at $t = 0$, $x(0) = 0$, and $v(0) = v_i$.

15.7a, but shifted to the right by one-fourth of a cycle. This shift is described mathematically by the phase constant $\phi = -\pi/2$, which is one-fourth of a full cycle of 2π.

Analysis Model   **Particle in Simple Harmonic Motion**

Imagine an object that is subject to a force that is proportional to the negative of the object's position, $F = -kx$. Such a force equation is known as Hooke's law, and it describes the force applied to an object attached to an ideal spring. The parameter k in Hooke's law is called the *spring constant* or the *force constant*. The position of an object acted on by a force described by Hooke's law is given by

$$x(t) = A \cos(\omega t + \phi) \tag{15.6}$$
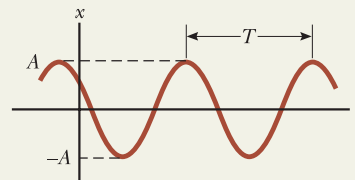
where A is the **amplitude** of the motion, ω is the **angular frequency,** and ϕ is the **phase constant.** The values of A and ϕ depend on the initial position and initial velocity of the particle.

The **period** of the oscillation of the particle is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{15.13}$$

and the inverse of the period is the **frequency.**

Examples:

- a bungee jumper hangs from a bungee cord and oscillates up and down
- a guitar string vibrates back and forth in a standing wave, with each element of the string moving in simple harmonic motion (Chapter 18)
- a piston in a gasoline engine oscillates up and down within the cylinder of the engine (Chapter 22)
- an atom in a diatomic molecule vibrates back and forth as if it is connected by a spring to the other atom in the molecule (Chapter 43)

Example 15.1   **A Block–Spring System**   AM

A 200-g block connected to a light spring for which the force constant is 5.00 N/m is free to oscillate on a frictionless, horizontal surface. The block is displaced 5.00 cm from equilibrium and released from rest as in Figure 15.6.

(A) Find the period of its motion.

SOLUTION

Conceptualize Study Figure 15.6 and imagine the block moving back and forth in simple harmonic motion once it is released. Set up an experimental model in the vertical direction by hanging a heavy object such as a stapler from a strong rubber band.

Categorize The block is modeled as a *particle in simple harmonic motion.*

. . . .

Analyze

Use Equation 15.9 to find the angular frequency of the block–spring system:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{5.00 \text{ N/m}}{200 \times 10^{-3} \text{ kg}}} = 5.00 \text{ rad/s}$$

Use Equation 15.13 to find the period of the system:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{5.00 \text{ rad/s}} = \boxed{1.26 \text{ s}}$$

(B) Determine the maximum speed of the block.

SOLUTION

Use Equation 15.17 to find v_{max}:

$$v_{max} = \omega A = (5.00 \text{ rad/s})(5.00 \times 10^{-2} \text{ m}) = \boxed{0.250 \text{ m/s}}$$

(C) What is the maximum acceleration of the block?

▶ **15.1** continued

SOLUTION

Use Equation 15.18 to find a_{max}: $\quad\quad\quad a_{max} = \omega^2 A = (5.00 \text{ rad/s})^2 (5.00 \times 10^{-2} \text{ m}) = \boxed{1.25 \text{ m/s}^2}$

(D) Express the position, velocity, and acceleration as functions of time in SI units.

SOLUTION

Find the phase constant from the initial condition that $\quad\quad x(0) = A \cos \phi = A \rightarrow \phi = 0$
$x = A$ at $t = 0$:

Use Equation 15.6 to write an expression for $x(t)$: $\quad\quad x = A \cos(\omega t + \phi) = \boxed{0.050\ 0 \cos 5.00t}$

Use Equation 15.15 to write an expression for $v(t)$: $\quad\quad v = -\omega A \sin(\omega t + \phi) = \boxed{-0.250 \sin 5.00t}$

Use Equation 15.16 to write an expression for $a(t)$: $\quad\quad a = -\omega^2 A \cos(\omega t + \phi) = \boxed{-1.25 \cos 5.00t}$

Finalize Consider part (a) of Figure 15.7, which shows the graphical representations of the motion of the block in this problem. Make sure that the mathematical representations found above in part (D) are consistent with these graphical representations.

WHAT IF? What if the block were released from the same initial position, $x_i = 5.00$ cm, but with an initial velocity of $v_i = -0.100$ m/s? Which parts of the solution change, and what are the new answers for those that do change?

Answers Part (A) does not change because the period is independent of how the oscillator is set into motion. Parts (B), (C), and (D) will change.

Write position and velocity expressions for the initial conditions:
$\quad$ (1) $\quad x(0) = A \cos \phi = x_i$

$\quad$ (2) $\quad v(0) = -\omega A \sin \phi = v_i$

Divide Equation (2) by Equation (1) to find the phase constant:
$$\frac{-\omega A \sin \phi}{A \cos \phi} = \frac{v_i}{x_i}$$

$$\tan \phi = -\frac{v_i}{\omega x_i} = -\frac{-0.100 \text{ m/s}}{(5.00 \text{ rad/s})(0.050\ 0 \text{ m})} = 0.400$$

$$\phi = \tan^{-1}(0.400) = 0.121\pi$$

Use Equation (1) to find A:
$$A = \frac{x_i}{\cos \phi} = \frac{0.050\ 0 \text{ m}}{\cos(0.121\pi)} = 0.053\ 9 \text{ m}$$

Find the new maximum speed:
$$v_{max} = \omega A = (5.00 \text{ rad/s})(5.39 \times 10^{-2} \text{ m}) = 0.269 \text{ m/s}$$

Find the new magnitude of the maximum acceleration:
$$a_{max} = \omega^2 A = (5.00 \text{ rad/s})^2 (5.39 \times 10^{-2} \text{ m}) = 1.35 \text{ m/s}^2$$

Find new expressions for position, velocity, and acceleration in SI units:
$$x = 0.053\ 9 \cos(5.00t + 0.121\pi)$$
$$v = -0.269 \sin(5.00t + 0.121\pi)$$
$$a = -1.35 \cos(5.00t + 0.121\pi)$$

As we saw in Chapters 7 and 8, many problems are easier to solve using an energy approach rather than one based on variables of motion. This particular What If? is easier to solve from an energy approach. Therefore, we shall investigate the energy of the simple harmonic oscillator in the next section.

Example 15.2 Watch Out for Potholes! AM

A car with a mass of 1 300 kg is constructed so that its frame is supported by four springs. Each spring has a force constant of 20 000 N/m. Two people riding in the car have a combined mass of 160 kg. Find the frequency of vibration of the car after it is driven over a pothole in the road.

▶ **15.2** continued

S O L U T I O N

Conceptualize Think about your experiences with automobiles. When you sit in a car, it moves downward a small distance because your weight is compressing the springs further. If you push down on the front bumper and release it, the front of the car oscillates a few times.

Categorize We imagine the car as being supported by a single spring and model the car as a *particle in simple harmonic motion.*

....

Analyze First, let's determine the effective spring constant of the four springs combined. For a given extension x of the springs, the combined force on the car is the sum of the forces from the individual springs.

Find an expression for the total force on the car:

$$F_{total} = \sum (-kx) = -\left(\sum k\right) x$$

In this expression, x has been factored from the sum because it is the same for all four springs. The effective spring constant for the combined springs is the sum of the individual spring constants.

Evaluate the effective spring constant:

$$k_{eff} = \sum k = 4 \times 20\,000 \text{ N/m} = 80\,000 \text{ N/m}$$

Use Equation 15.14 to find the frequency of vibration:

$$f = \frac{1}{2\pi} \sqrt{\frac{k_{eff}}{m}} = \frac{1}{2\pi} \sqrt{\frac{80\,000 \text{ N/m}}{1\,460 \text{ kg}}} = \boxed{1.18 \text{ Hz}}$$

....

Finalize The mass we used here is that of the car plus the people because that is the total mass that is oscillating. Also notice that we have explored only up-and-down motion of the car. If an oscillation is established in which the car rocks back and forth such that the front end goes up when the back end goes down, the frequency will be different.

WHAT IF? Suppose the car stops on the side of the road and the two people exit the car. One of them pushes downward on the car and releases it so that it oscillates vertically. Is the frequency of the oscillation the same as the value we just calculated?

Answer The suspension system of the car is the same, but the mass that is oscillating is smaller: it no longer includes the mass of the two people. Therefore, the frequency should be higher. Let's calculate the new frequency, taking the mass to be 1 300 kg:

$$f = \frac{1}{2\pi} \sqrt{\frac{k_{eff}}{m}} = \frac{1}{2\pi} \sqrt{\frac{80\,000 \text{ N/m}}{1\,300 \text{ kg}}} = 1.25 \text{ Hz}$$

As predicted, the new frequency is a bit higher.

15.3 Energy of the Simple Harmonic Oscillator

As we have done before, after studying the the motion of an object modeled as a particle in a new situation and investigating the forces involved in influencing that motion, we turn our attention to *energy*. Let us examine the mechanical energy of a system in which a particle undergoes simple harmonic motion, such as the block–spring system illustrated in Figure 15.1. Because the surface is frictionless, the system is isolated and we expect the total mechanical energy of the system to be constant. We assume a massless spring, so the kinetic energy of the system corresponds only to that of the block. We can use Equation 15.15 to express the kinetic energy of the block as
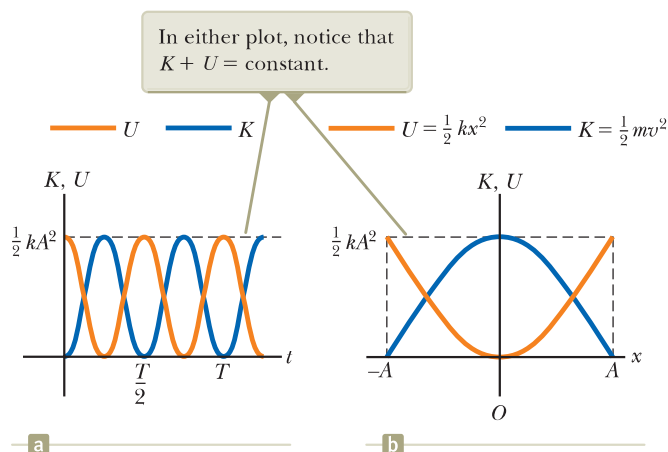
Kinetic energy of a simple ▶ harmonic oscillator

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) \qquad (15.19)$$

The elastic potential energy stored in the spring for any elongation x is given by $\tfrac{1}{2}kx^2$ (see Eq. 7.22). Using Equation 15.6 gives

Potential energy of a simple ▶ harmonic oscillator

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2 \cos^2(\omega t + \phi) \qquad (15.20)$$

We see that K and U are *always* positive quantities or zero. Because $\omega^2 = k/m$, we can express the total mechanical energy of the simple harmonic oscillator as

$$E = K + U = \tfrac{1}{2}kA^2[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)]$$

From the identity $\sin^2\theta + \cos^2\theta = 1$, we see that the quantity in square brackets is unity. Therefore, this equation reduces to

$$E = \tfrac{1}{2}kA^2 \tag{15.21}$$

◀ **Total energy of a simple harmonic oscillator**

That is, the total mechanical energy of a simple harmonic oscillator is a constant of the motion and is proportional to the square of the amplitude. The total mechanical energy is equal to the maximum potential energy stored in the spring when $x = \pm A$ because $v = 0$ at these points and there is no kinetic energy. At the equilibrium position, where $U = 0$ because $x = 0$, the total energy, all in the form of kinetic energy, is again $\tfrac{1}{2}kA^2$.

Plots of the kinetic and potential energies versus time appear in Figure 15.9a, where we have taken $\phi = 0$. At all times, the sum of the kinetic and potential energies is a constant equal to $\tfrac{1}{2}kA^2$, the total energy of the system.

The variations of K and U with the position x of the block are plotted in Figure 15.9b. Energy is continuously being transformed between potential energy stored in the spring and kinetic energy of the block.

Figure 15.10 on page 460 illustrates the position, velocity, acceleration, kinetic energy, and potential energy of the block–spring system for one full period of the motion. Most of the ideas discussed so far are incorporated in this important figure. Study it carefully.

Finally, we can obtain the velocity of the block at an arbitrary position by expressing the total energy of the system at some arbitrary position x as
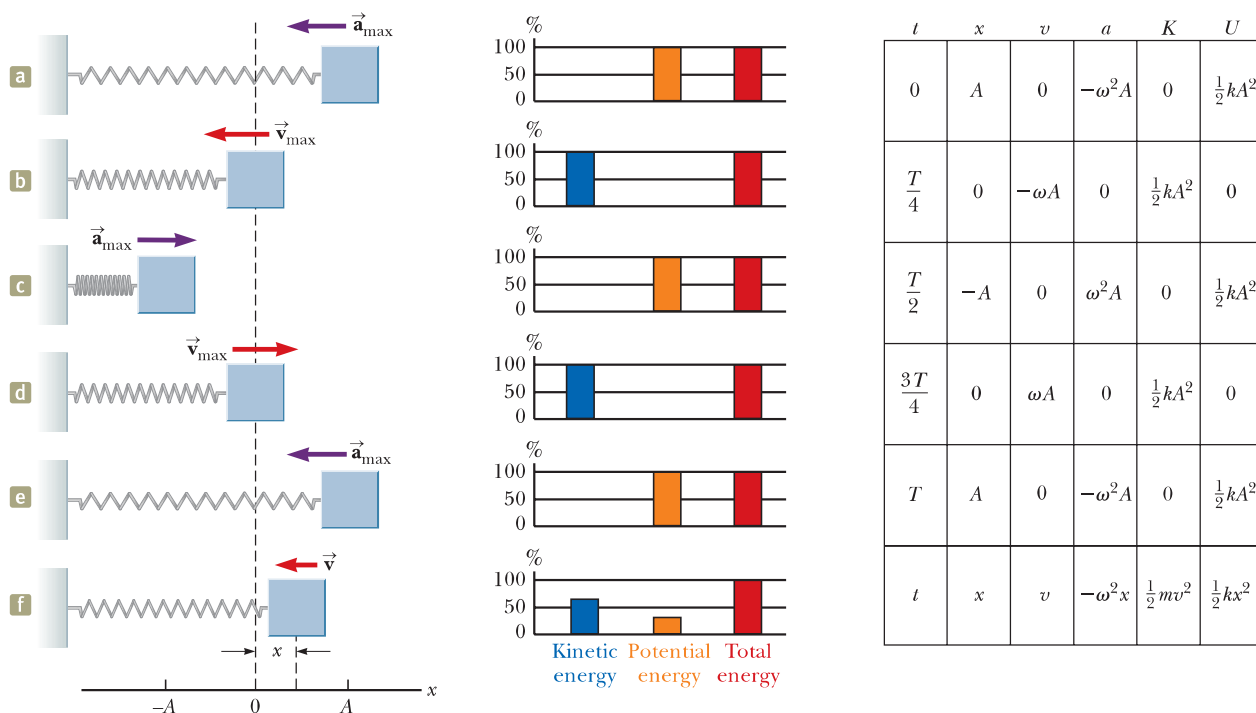
$$E = K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2$$

$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} = \pm\omega\sqrt{A^2 - x^2} \tag{15.22}$$

◀ **Velocity as a function of position for a simple harmonic oscillator**

When you check Equation 15.22 to see whether it agrees with known cases, you find that it verifies that the speed is a maximum at $x = 0$ and is zero at the turning points $x = \pm A$.

You may wonder why we are spending so much time studying simple harmonic oscillators. We do so because they are good models of a wide variety of physical phenomena. For example, recall the Lennard–Jones potential discussed in Example 7.9. This complicated function describes the forces holding atoms together. Figure 15.11a on page 460 shows that for small displacements from the equilibrium

| t	x	v	a	K	U
0	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$
$\frac{T}{4}$	0	$-\omega A$	0	$\frac{1}{2}kA^2$	0
$\frac{T}{2}$	$-A$	0	$\omega^2 A$	0	$\frac{1}{2}kA^2$
$\frac{3T}{4}$	0	ωA	0	$\frac{1}{2}kA^2$	0
T	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$
t	x	v	$-\omega^2 x$	$\frac{1}{2}mv^2$	$\frac{1}{2}kx^2$

Figure 15.10 (a) through (e) Several instants in the simple harmonic motion for a block–spring system. Energy bar graphs show the distribution of the energy of the system at each instant. The parameters in the table at the right refer to the block–spring system, assuming at $t = 0$, $x = A$; hence, $x = A \cos \omega t$. For these five special instants, one of the types of energy is zero. (f) An arbitrary point in the motion of the oscillator. The system possesses both kinetic energy and potential energy at this instant as shown in the bar graph.

position, the potential energy curve for this function approximates a parabola, which represents the potential energy function for a simple harmonic oscillator. Therefore, we can model the complex atomic binding forces as being due to tiny springs as depicted in Figure 15.11b.

The ideas presented in this chapter apply not only to block–spring systems and atoms, but also to a wide range of situations that include bungee jumping, playing a musical instrument, and viewing the light emitted by a laser. You will see more examples of simple harmonic oscillators as you work through this book.

Figure 15.11 (a) If the atoms in a molecule do not move too far from their equilibrium positions, a graph of potential energy versus separation distance between atoms is similar to the graph of potential energy versus position for a simple harmonic oscillator (dashed black curve). (b) The forces between atoms in a solid can be modeled by imagining springs between neighboring atoms.
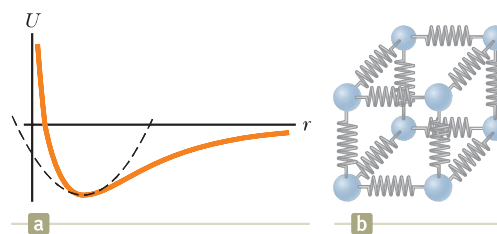
Example 15.3   Oscillations on a Horizontal Surface   AM

A 0.500-kg cart connected to a light spring for which the force constant is 20.0 N/m oscillates on a frictionless, horizontal air track.

(A) Calculate the maximum speed of the cart if the amplitude of the motion is 3.00 cm.

SOLUTION

Conceptualize The system oscillates in exactly the same way as the block in Figure 15.10, so use that figure in your mental image of the motion.

▶ **15.3** continued

Categorize The cart is modeled as a *particle in simple harmonic motion.*

Analyze Use Equation 15.21 to express the total energy of the oscillator system and equate it to the kinetic energy of the system when the cart is at $x = 0$:

$$E = \tfrac{1}{2}kA^2 = \tfrac{1}{2}mv_{max}^2$$

Solve for the maximum speed and substitute numerical values:

$$v_{max} = \sqrt{\frac{k}{m}}\,A = \sqrt{\frac{20.0\ \text{N/m}}{0.500\ \text{kg}}}(0.030\ 0\ \text{m}) = \boxed{0.190\ \text{m/s}}$$

(B) What is the velocity of the cart when the position is 2.00 cm?

SOLUTION

Use Equation 15.22 to evaluate the velocity:

$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)}$$

$$= \pm\sqrt{\frac{20.0\ \text{N/m}}{0.500\ \text{kg}}[(0.030\ 0\ \text{m})^2 - (0.020\ 0\ \text{m})^2]}$$

$$= \boxed{\pm 0.141\ \text{m/s}}$$

The positive and negative signs indicate that the cart could be moving to either the right or the left at this instant.

(C) Compute the kinetic and potential energies of the system when the position of the cart is 2.00 cm.

SOLUTION

Use the result of part (B) to evaluate the kinetic energy at $x = 0.020\ 0$ m:

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(0.500\ \text{kg})(0.141\ \text{m/s})^2 = \boxed{5.00 \times 10^{-3}\ \text{J}}$$

Evaluate the elastic potential energy at $x = 0.020\ 0$ m:

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}(20.0\ \text{N/m})(0.0200\ \text{m})^2 = \boxed{4.00 \times 10^{-3}\ \text{J}}$$

Finalize The sum of the kinetic and potential energies in part (C) is equal to the total energy, which can be found from Equation 15.21. That must be true for *any* position of the cart.

WHAT IF?   The cart in this example could have been set into motion by releasing the cart from rest at $x = 3.00$ cm. What if the cart were released from the same position, but with an initial velocity of $v = -0.100$ m/s? What are the new amplitude and maximum speed of the cart?

Answer This question is of the same type we asked at the end of Example 15.1, but here we apply an energy approach.

First calculate the total energy of the system at $t = 0$:

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$

$$= \tfrac{1}{2}(0.500\ \text{kg})(-0.100\ \text{m/s})^2 + \tfrac{1}{2}(20.0\ \text{N/m})(0.030\ 0\ \text{m})^2$$

$$= 1.15 \times 10^{-2}\ \text{J}$$

Equate this total energy to the potential energy of the system when the cart is at the endpoint of the motion:

$$E = \tfrac{1}{2}kA^2$$

Solve for the amplitude A:

$$A = \sqrt{\frac{2E}{k}} = \sqrt{\frac{2(1.15 \times 10^{-2}\ \text{J})}{20.0\ \text{N/m}}} = 0.033\ 9\ \text{m}$$

Equate the total energy to the kinetic energy of the system when the cart is at the equilibrium position:

$$E = \tfrac{1}{2}mv_{max}^2$$

Solve for the maximum speed:

$$v_{max} = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2(1.15 \times 10^{-2}\ \text{J})}{0.500\ \text{kg}}} = 0.214\ \text{m/s}$$

The amplitude and maximum velocity are larger than the previous values because the cart was given an initial velocity at $t = 0$.

The back edge of the treadle goes up and down as one's feet rock the treadle.

The oscillation of the treadle causes circular motion of the drive wheel, eventually resulting in additional up and down motion—of the sewing needle.

John W. Jewett, Jr.

Figure 15.12 The bottom of a treadle-style sewing machine from the early twentieth century. The treadle is the wide, flat foot pedal with the metal grillwork.

15.4 Comparing Simple Harmonic Motion with Uniform Circular Motion

Some common devices in everyday life exhibit a relationship between oscillatory motion and circular motion. For example, consider the drive mechanism for a non-electric sewing machine in Figure 15.12. The operator of the machine places her feet on the treadle and rocks them back and forth. This oscillatory motion causes the large wheel at the right to undergo circular motion. The red drive belt seen in the photograph transfers this circular motion to the sewing machine mechanism (above the photo) and eventually results in the oscillatory motion of the sewing needle. In this section, we explore this interesting relationship between these two types of motion.

Figure 15.13 is a view of an experimental arrangement that shows this relationship. A ball is attached to the rim of a turntable of radius A, which is illuminated from above by a lamp. The ball casts a shadow on a screen. As the turntable rotates with constant angular speed, the shadow of the ball moves back and forth in simple harmonic motion.

Consider a particle located at point P on the circumference of a circle of radius A as in Figure 15.14a, with the line OP making an angle ϕ with the x axis at $t = 0$. We call this circle a *reference circle* for comparing simple harmonic motion with uniform circular motion, and we choose the position of P at $t = 0$ as our reference position. If the particle moves along the circle with constant angular speed ω until OP makes an angle θ with the x axis as in Figure 15.14b, at some time $t > 0$ the angle between OP and the x axis is $\theta = \omega t + \phi$. As the particle moves along the circle, the projection of P on the x axis, labeled point Q, moves back and forth along the x axis between the limits $x = \pm A$.

Notice that points P and Q always have the same x coordinate. From the right triangle OPQ, we see that this x coordinate is
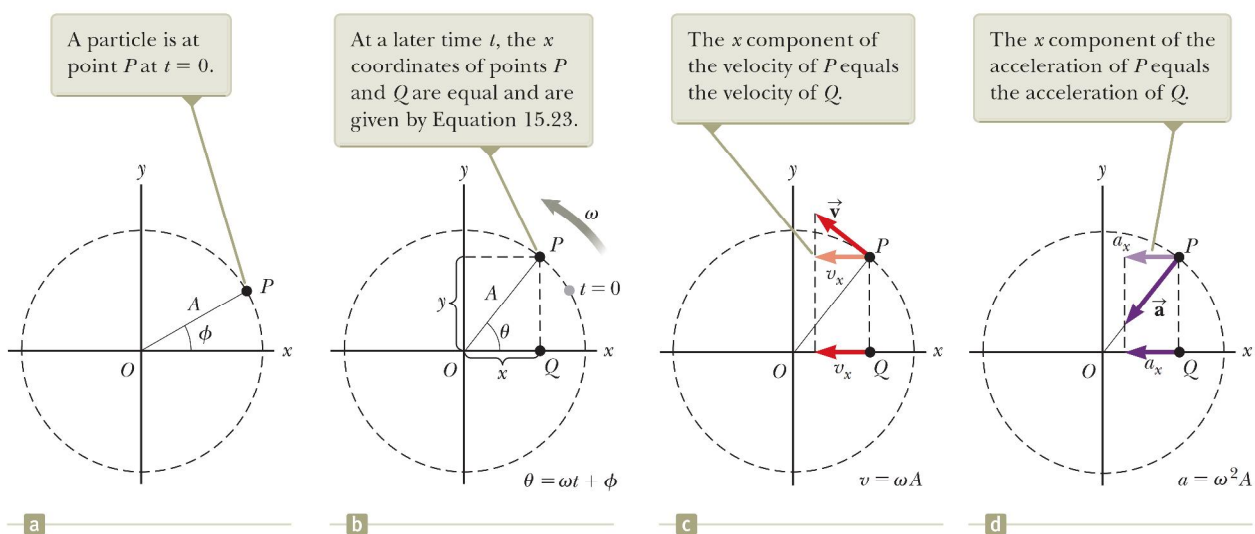
$$x(t) = A \cos(\omega t + \phi) \tag{15.23}$$

This expression is the same as Equation 15.6 and shows that the point Q moves with simple harmonic motion along the x axis. Therefore, the motion of an object described by the analysis model of a particle in simple harmonic motion along a straight line can be represented by the projection of an object that can be modeled as a particle in uniform circular motion along a diameter of a reference circle.

This geometric interpretation shows that the time interval for one complete revolution of the point P on the reference circle is equal to the period of motion T for simple harmonic motion between $x = \pm A$. Therefore, the angular speed ω of P is the same as the angular frequency ω of simple harmonic motion along the x axis

The ball rotates like a particle in uniform circular motion.

Lamp

A

Turntable

A Screen

The ball's shadow moves like a particle in simple harmonic motion.
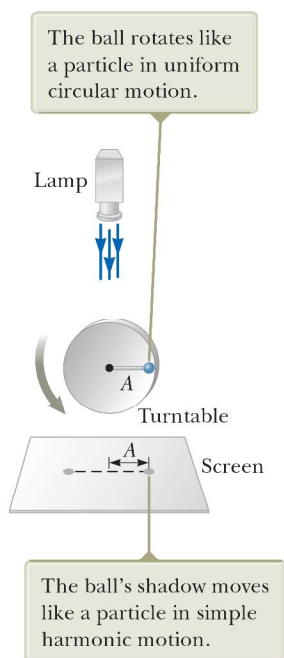
Figure 15.13 An experimental setup for demonstrating the connection between a particle in simple harmonic motion and a corresponding particle in uniform circular motion.

A particle is at point P at $t = 0$.

At a later time t, the x coordinates of points P and Q are equal and are given by Equation 15.23.

The x component of the velocity of P equals the velocity of Q.

The x component of the acceleration of P equals the acceleration of Q.

$\theta = \omega t + \phi$

$v = \omega A$

$a = \omega^2 A$

Figure 15.14 Relationship between the uniform circular motion of a point P and the simple harmonic motion of a point Q. A particle at P moves in a circle of radius A with constant angular speed ω.

(which is why we use the same symbol). The phase constant ϕ for simple harmonic motion corresponds to the initial angle OP makes with the x axis. The radius A of the reference circle equals the amplitude of the simple harmonic motion.

Because the relationship between linear and angular speed for circular motion is $v = r\omega$ (see Eq. 10.10), the particle moving on the reference circle of radius A has a velocity of magnitude ωA. From the geometry in Figure 15.14c, we see that the x component of this velocity is $-\omega A \sin(\omega t + \phi)$. By definition, point Q has a velocity given by dx/dt. Differentiating Equation 15.23 with respect to time, we find that the velocity of Q is the same as the x component of the velocity of P.

The acceleration of P on the reference circle is directed radially inward toward O and has a magnitude $v^2/A = \omega^2 A$. From the geometry in Figure 15.14d, we see that the x component of this acceleration is $-\omega^2 A \cos(\omega t + \phi)$. This value is also the acceleration of the projected point Q along the x axis, as you can verify by taking the second derivative of Equation 15.23.

Quick Quiz 15.5 Figure 15.15 shows the position of an object in uniform circular motion at $t = 0$. A light shines from above and projects a shadow of the object on a screen below the circular motion. What are the correct values for the *amplitude* and *phase constant* (relative to an x axis to the right) of the simple harmonic motion of the shadow? **(a)** 0.50 m and 0 **(b)** 1.00 m and 0 **(c)** 0.50 m and π **(d)** 1.00 m and π
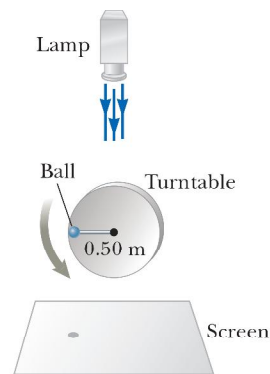
Figure 15.15 (Quick Quiz 15.5) An object moves in circular motion, casting a shadow on the screen below. Its position at an instant of time is shown.

Example 15.4   **Circular Motion with Constant Angular Speed**   **AM**

The ball in Figure 15.13 rotates counterclockwise in a circle of radius 3.00 m with a constant angular speed of 8.00 rad/s. At $t = 0$, its shadow has an x coordinate of 2.00 m and is moving to the right.

(A) Determine the x coordinate of the shadow as a function of time in SI units.

SOLUTION

Conceptualize Be sure you understand the relationship between circular motion of the ball and simple harmonic motion of its shadow as described in Figure 15.13. Notice that the shadow is *not* at is maximum position at $t = 0$.

Categorize The ball on the turntable is a *particle in uniform circular motion*. The shadow is modeled as a *particle in simple harmonic motion*.

▶ **15.4** continued

....

Analyze Use Equation 15.23 to write an expression for the x coordinate of the rotating ball:

$$x = A\cos(\omega t + \phi)$$

Solve for the phase constant:

$$\phi = \cos^{-1}\left(\frac{x}{A}\right) - \omega t$$

Substitute numerical values for the initial conditions:

$$\phi = \cos^{-1}\left(\frac{2.00\ \text{m}}{3.00\ \text{m}}\right) - 0 = \pm48.2° = \pm0.841\ \text{rad}$$

If we were to take $\phi = +0.841$ rad as our answer, the shadow would be moving to the left at $t = 0$. Because the shadow is moving to the right at $t = 0$, we must choose $\phi = -0.841$ rad.

Write the x coordinate as a function of time:

$$x = \boxed{3.00\cos(8.00t - 0.841)}$$

(B) Find the x components of the shadow's velocity and acceleration at any time t.

SOLUTION

Differentiate the x coordinate with respect to time to find the velocity at any time in m/s:

$$v_x = \frac{dx}{dt} = (-3.00\ \text{m})(8.00\ \text{rad/s})\sin(8.00t - 0.841)$$

$$= \boxed{-24.0\sin(8.00t - 0.841)}$$

Differentiate the velocity with respect to time to find the acceleration at any time in m/s²:

$$a_x = \frac{dv_x}{dt} = (-24.0\ \text{m/s})(8.00\ \text{rad/s})\cos(8.00t - 0.841)$$

$$= \boxed{-192\cos(8.00t - 0.841)}$$

Finalize These results are equally valid for the ball moving in uniform circular motion and the shadow moving in simple harmonic motion. Notice that the value of the phase constant puts the ball in the fourth quadrant of the xy coordinate system of Figure 15.14, which is consistent with the shadow having a positive value for x and moving toward the right.

15.5 The Pendulum

When θ is small, a simple pendulum's motion can be modeled as simple harmonic motion about the equilibrium position $\theta = 0$.
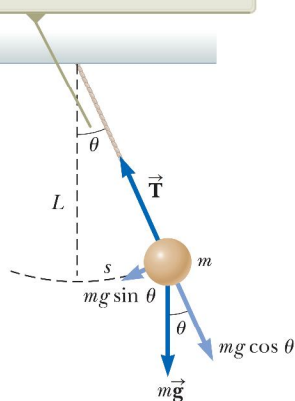
Figure 15.16 A simple pendulum.

The **simple pendulum** is another mechanical system that exhibits periodic motion. It consists of a particle-like bob of mass m suspended by a light string of length L that is fixed at the upper end as shown in Figure 15.16. The motion occurs in the vertical plane and is driven by the gravitational force. We shall show that, provided the angle θ is small (less than about 10°), the motion is very close to that of a simple harmonic oscillator.

The forces acting on the bob are the force $\vec{T}$ exerted by the string and the gravitational force $m\vec{g}$. The tangential component $mg\sin\theta$ of the gravitational force always acts toward $\theta = 0$, opposite the displacement of the bob from the lowest position. Therefore, the tangential component is a restoring force, and we can apply Newton's second law for motion in the tangential direction:

$$F_t = ma_t \quad \rightarrow \quad -mg\sin\theta = m\frac{d^2s}{dt^2}$$

where the negative sign indicates that the tangential force acts toward the equilibrium (vertical) position and s is the bob's position measured along the arc. We have expressed the tangential acceleration as the second derivative of the position s. Because $s = L\theta$ (Eq. 10.1a with $r = L$) and L is constant, this equation reduces to

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

Considering θ as the position, let us compare this equation with Equation 15.3. Does it have the same mathematical form? No! The right side is proportional to $\sin \theta$ rather than to θ; hence, we would not expect simple harmonic motion because this expression is not of the same mathematical form as Equation 15.3. If we assume θ is *small* (less than about 10° or 0.2 rad), however, we can use the **small angle approximation,** in which $\sin \theta \approx \theta$, where θ is measured in radians. Table 15.1 shows angles in degrees and radians and the sines of these angles. As long as θ is less than approximately 10°, the angle in radians and its sine are the same to within an accuracy of less than 1.0%.

Therefore, for small angles, the equation of motion becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta \quad \text{(for small values of }\theta\text{)} \tag{15.24}$$

Equation 15.24 has the same mathematical form as Equation 15.3, so we conclude that the motion for small amplitudes of oscillation can be modeled as simple harmonic motion. Therefore, the solution of Equation 15.24 is modeled after Equation 15.6 and is given by $\theta = \theta_{max} \cos(\omega t + \phi)$, where θ_{max} is the *maximum angular position* and the angular frequency ω is

$$\omega = \sqrt{\frac{g}{L}} \tag{15.25}$$

◀ **Angular frequency for a simple pendulum**

The period of the motion is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \tag{15.26}$$

◀ **Period of a simple pendulum**

In other words, the period and frequency of a simple pendulum depend only on the length of the string and the acceleration due to gravity. Because the period is independent of the mass, we conclude that all simple pendula that are of equal length and are at the same location (so that g is constant) oscillate with the same period.

The simple pendulum can be used as a timekeeper because its period depends only on its length and the local value of g. It is also a convenient device for making precise measurements of the free-fall acceleration. Such measurements are important because variations in local values of g can provide information on the location of oil and other valuable underground resources.

Quick Quiz 15.6 A grandfather clock depends on the period of a pendulum to keep correct time. **(i)** Suppose a grandfather clock is calibrated correctly and then a mischievous child slides the bob of the pendulum downward on the oscillating rod. Does the grandfather clock run (a) slow, (b) fast, or (c) correctly? **(ii)** Suppose a grandfather clock is calibrated correctly at sea level and is then taken to the top of a very tall mountain. Does the grandfather clock now run (a) slow, (b) fast, or (c) correctly?

Table 15.1   Angles and Sines of Angles

Angle in Degrees	Angle in Radians	Sine of Angle	Percent Difference
0°	0.000 0	0.000 0	0.0%
1°	0.017 5	0.017 5	0.0%
2°	0.034 9	0.034 9	0.0%
3°	0.052 4	0.052 3	0.0%
5°	0.087 3	0.087 2	0.1%
10°	0.174 5	0.173 6	0.5%
15°	0.261 8	0.258 8	1.2%
20°	0.349 1	0.342 0	2.1%
30°	0.523 6	0.500 0	4.7%

Example 15.5 | A Connection Between Length and Time

Christian Huygens (1629–1695), the greatest clockmaker in history, suggested that an international unit of length could be defined as the length of a simple pendulum having a period of exactly 1 s. How much shorter would our length unit be if his suggestion had been followed?

SOLUTION

Conceptualize Imagine a pendulum that swings back and forth in exactly 1 second. Based on your experience in observing swinging objects, can you make an estimate of the required length? Hang a small object from a string and simulate the 1-s pendulum.

Categorize This example involves a simple pendulum, so we categorize it as a substitution problem that applies the concepts introduced in this section.

Solve Equation 15.26 for the length and substitute the known values:

$$L = \frac{T^2 g}{4\pi^2} = \frac{(1.00 \text{ s})^2 (9.80 \text{ m/s}^2)}{4\pi^2} = \boxed{0.248 \text{ m}}$$

The meter's length would be slightly less than one-fourth of its current length. Also, the number of significant digits depends only on how precisely we know g because the time has been defined to be exactly 1 s.

WHAT IF? What if Huygens had been born on another planet? What would the value for g have to be on that planet such that the meter based on Huygens's pendulum would have the same value as our meter?

Answer Solve Equation 15.26 for g:

$$g = \frac{4\pi^2 L}{T^2} = \frac{4\pi^2 (1.00 \text{ m})}{(1.00 \text{ s})^2} = 4\pi^2 \text{ m/s}^2 = 39.5 \text{ m/s}^2$$

No planet in our solar system has an acceleration due to gravity that large.
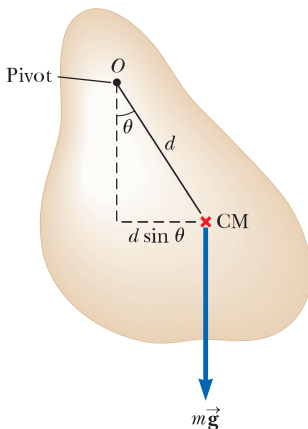
Physical Pendulum

Figure 15.17 A physical pendulum pivoted at O.

Suppose you balance a wire coat hanger so that the hook is supported by your extended index finger. When you give the hanger a small angular displacement with your other hand and then release it, it oscillates. If a hanging object oscillates about a fixed axis that does not pass through its center of mass and the object cannot be approximated as a point mass, we cannot treat the system as a simple pendulum. In this case, the system is called a **physical pendulum.**

Consider a rigid object pivoted at a point O that is a distance d from the center of mass (Fig. 15.17). The gravitational force provides a torque about an axis through O, and the magnitude of that torque is $mgd \sin \theta$, where θ is as shown in Figure 15.17. We apply the rigid object under a net torque analysis model to the object and use the rotational form of Newton's second law, $\Sigma \tau_{\text{ext}} = I\alpha$, where I is the moment of inertia of the object about the axis through O. The result is

$$-mgd \sin \theta = I \frac{d^2\theta}{dt^2}$$

The negative sign indicates that the torque about O tends to decrease θ. That is, the gravitational force produces a restoring torque. If we again assume θ is small, the approximation $\sin \theta \approx \theta$ is valid and the equation of motion reduces to

$$\frac{d^2\theta}{dt^2} = -\left(\frac{mgd}{I}\right)\theta = -\omega^2\theta \qquad \text{(15.27)}$$

Because this equation is of the same mathematical form as Equation 15.3, its solution is modeled after that of the simple harmonic oscillator. That is, the solution

of Equation 15.27 is given by $\theta = \theta_{max} \cos(\omega t + \phi)$, where θ_{max} is the maximum angular position and

$$\omega = \sqrt{\frac{mgd}{I}}$$

The period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}$$ (15.28)   ◀ Period of a physical pendulum

This result can be used to measure the moment of inertia of a flat, rigid object. If the location of the center of mass—and hence the value of d—is known, the moment of inertia can be obtained by measuring the period. Finally, notice that Equation 15.28 reduces to the period of a simple pendulum (Eq. 15.26) when $I = md^2$, that is, when all the mass is concentrated at the center of mass.

Example 15.6   A Swinging Rod

A uniform rod of mass M and length L is pivoted about one end and oscillates in a vertical plane (Fig. 15.18). Find the period of oscillation if the amplitude of the motion is small.

SOLUTION

Conceptualize Imagine a rod swinging back and forth when pivoted at one end. Try it with a meterstick or a scrap piece of wood.

Categorize Because the rod is not a point particle, we categorize it as a physical pendulum.

Figure 15.18 (Example 15.6) A rigid rod oscillating about a pivot through one end is a physical pendulum with $d = L/2$.

Analyze In Chapter 10, we found that the moment of inertia of a uniform rod about an axis through one end is $\frac{1}{3}ML^2$. The distance d from the pivot to the center of mass of the rod is $L/2$.

Substitute these quantities into Equation 15.28:

$$T = 2\pi\sqrt{\frac{\frac{1}{3}ML^2}{Mg(L/2)}} = 2\pi\sqrt{\frac{2L}{3g}}$$

Finalize In one of the Moon landings, an astronaut walking on the Moon's surface had a belt hanging from his space suit, and the belt oscillated as a physical pendulum. A scientist on the Earth observed this motion on television and used it to estimate the free-fall acceleration on the Moon. How did the scientist make this calculation?
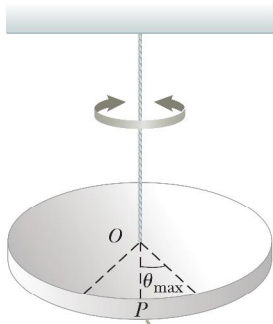

Torsional Pendulum

Figure 15.19 on page 468 shows a rigid object such as a disk suspended by a wire attached at the top to a fixed support. When the object is twisted through some angle θ, the twisted wire exerts on the object a restoring torque that is proportional to the angular position. That is,
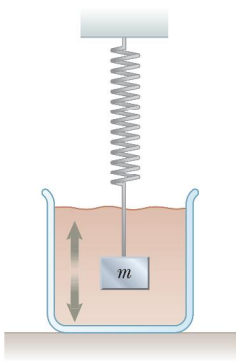
$$\tau = -\kappa\theta$$

where κ (Greek letter kappa) is called the *torsion constant* of the support wire and is a rotational analog to the force constant k for a spring. The value of κ can be obtained by applying a known torque to twist the wire through a measurable angle θ. Applying Newton's second law for rotational motion, we find that

$$\sum \tau = I\alpha \quad \rightarrow \quad -\kappa\theta = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta \qquad (15.29)$$

Again, this result is the equation of motion for a simple harmonic oscillator, with $\omega = \sqrt{\kappa/I}$ and a period

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \qquad (15.30)$$

This system is called a *torsional pendulum*. There is no small-angle restriction in this situation as long as the elastic limit of the wire is not exceeded.

The object oscillates about the line *OP* with an amplitude θ_{max}.

Figure 15.19 A torsional pendulum.

15.6 Damped Oscillations

The oscillatory motions we have considered so far have been for ideal systems, that is, systems that oscillate indefinitely under the action of only one force, a linear restoring force. In many real systems, nonconservative forces such as friction or air resistance also act and retard the motion of the system. Consequently, the mechanical energy of the system diminishes in time, and the motion is said to be *damped*. The mechanical energy of the system is transformed into internal energy in the object and the retarding medium. Figure 15.20 depicts one such system: an object attached to a spring and submersed in a viscous liquid. Another example is a simple pendulum oscillating in air. After being set into motion, the pendulum eventually stops oscillating due to air resistance. The opening photograph for this chapter depicts damped oscillations in practice. The spring-loaded devices mounted below the bridge are dampers that transform mechanical energy of the oscillating bridge into internal energy.

One common type of retarding force is that discussed in Section 6.4, where the force is proportional to the speed of the moving object and acts in the direction opposite the velocity of the object with respect to the medium. This retarding force is often observed when an object moves through air, for instance. Because the retarding force can be expressed as $\vec{R} = -b\vec{v}$ (where b is a constant called the *damping coefficient*) and the restoring force of the system is $-kx$, we can write Newton's second law as

Figure 15.20 One example of a damped oscillator is an object attached to a spring and submersed in a viscous liquid.

$$\sum F_x = -kx - bv_x = ma_x$$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2} \qquad (15.31)$$

The solution to this equation requires mathematics that may be unfamiliar to you; we simply state it here without proof. When the retarding force is small compared with the maximum restoring force—that is, when the damping coefficient b is small—the solution to Equation 15.31 is

$$x = Ae^{-(b/2m)t}\cos(\omega t + \phi) \qquad (15.32)$$

where the angular frequency of oscillation is

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \qquad (15.33)$$

This result can be verified by substituting Equation 15.32 into Equation 15.31. It is convenient to express the angular frequency of a damped oscillator in the form

$$\omega = \sqrt{{\omega_0}^2 - \left(\frac{b}{2m}\right)^2}$$

where $\omega_0 = \sqrt{k/m}$ represents the angular frequency in the absence of a retarding force (the undamped oscillator) and is called the **natural frequency** of the system.

Figure 15.21 shows the position as a function of time for an object oscillating in the presence of a retarding force. When the retarding force is small, the oscillatory character of the motion is preserved but the amplitude decreases exponentially in time, with the result that the motion ultimately becomes undetectable. Any system that behaves in this way is known as a **damped oscillator.** The dashed black lines in Figure 15.21, which define the *envelope* of the oscillatory curve, represent the exponential factor in Equation 15.32. This envelope shows that the amplitude decays exponentially with time. For motion with a given spring constant and object mass, the oscillations dampen more rapidly for larger values of the retarding force.

When the magnitude of the retarding force is small such that $b/2m < \omega_0$, the system is said to be **underdamped.** The resulting motion is represented by Figure 15.21 and the the blue curve in Figure 15.22. As the value of b increases, the amplitude of the oscillations decreases more and more rapidly. When b reaches a critical value b_c such that $b_c/2m = \omega_0$, the system does not oscillate and is said to be **critically damped.** In this case, the system, once released from rest at some nonequilibrium position, approaches but does not pass through the equilibrium position. The graph of position versus time for this case is the red curve in Figure 15.22.

If the medium is so viscous that the retarding force is large compared with the restoring force—that is, if $b/2m > \omega_0$—the system is **overdamped.** Again, the displaced system, when free to move, does not oscillate but rather simply returns to its equilibrium position. As the damping increases, the time interval required for the system to approach equilibrium also increases as indicated by the black curve in Figure 15.22. For critically damped and overdamped systems, there is no angular frequency ω and the solution in Equation 15.32 is not valid.

Figure 15.21 Graph of position versus time for a damped oscillator.

Figure 15.22 Graphs of position versus time for an underdamped oscillator (blue curve), a critically damped oscillator (red curve), and an overdamped oscillator (black curve).
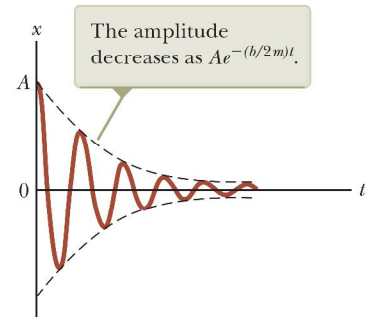
15.7 Forced Oscillations

We have seen that the mechanical energy of a damped oscillator decreases in time as a result of the retarding force. It is possible to compensate for this energy decrease by applying a periodic external force that does positive work on the system. At any instant, energy can be transferred into the system by an applied force that acts in the direction of motion of the oscillator. For example, a child on a swing can be kept in motion by appropriately timed "pushes." The amplitude of motion remains constant if the energy input per cycle of motion exactly equals the decrease in mechanical energy in each cycle that results from retarding forces.

A common example of a forced oscillator is a damped oscillator driven by an external force that varies periodically, such as $F(t) = F_0 \sin \omega t$, where F_0 is a constant and ω is the angular frequency of the driving force. In general, the frequency ω of the driving force is variable, whereas the natural frequency ω_0 of the oscillator is fixed by the values of k and m. Modeling an oscillator with both retarding and driving forces as a particle under a net force, Newton's second law in this situation gives
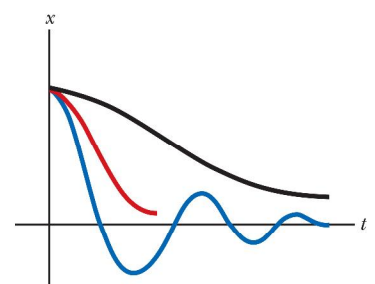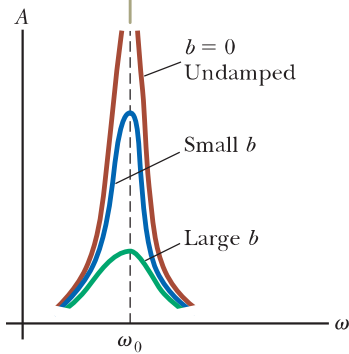
$$\sum F_x = ma_x \rightarrow F_0 \sin \omega t - b\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2} \quad \textbf{(15.34)}$$

Again, the solution of this equation is rather lengthy and will not be presented. After the driving force on an initially stationary object begins to act, the amplitude of the oscillation will increase. The system of the oscillator and the surrounding medium is a nonisolated system: work is done by the driving force, such that the vibrational energy of the system (kinetic energy of the object, elastic potential energy in the spring) and internal energy of the object and the medium increase. After a sufficiently long period of time, when the energy input per cycle from the driving force equals the amount of mechanical energy transformed to internal energy for each cycle, a steady-state condition is reached in which the oscillations proceed with constant amplitude. In this situation, the solution of Equation 15.34 is

$$x = A \cos(\omega t + \phi) \quad \textbf{(15.35)}$$

where

$$A = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0{}^2)^2 + \left(\dfrac{b\omega}{m}\right)^2}} \qquad (15.36)$$

and where $\omega_0 = \sqrt{k/m}$ is the natural frequency of the undamped oscillator ($b = 0$).

Equations 15.35 and 15.36 show that the forced oscillator vibrates at the frequency of the driving force and that the amplitude of the oscillator is constant for a given driving force because it is being driven in steady-state by an external force. For small damping, the amplitude is large when the frequency of the driving force is near the natural frequency of oscillation, or when $\omega \approx \omega_0$. The dramatic increase in amplitude near the natural frequency is called **resonance,** and the natural frequency ω_0 is also called the **resonance frequency** of the system.

The reason for large-amplitude oscillations at the resonance frequency is that energy is being transferred to the system under the most favorable conditions. We can better understand this concept by taking the first time derivative of x in Equation 15.35, which gives an expression for the velocity of the oscillator. We find that v is proportional to $\sin(\omega t + \phi)$, which is the same trigonometric function as that describing the driving force. Therefore, the applied force $\vec{\mathbf{F}}$ is in phase with the velocity. The rate at which work is done on the oscillator by $\vec{\mathbf{F}}$ equals the dot product $\vec{\mathbf{F}} \cdot \vec{\mathbf{v}}$; this rate is the power delivered to the oscillator. Because the product $\vec{\mathbf{F}} \cdot \vec{\mathbf{v}}$ is a maximum when $\vec{\mathbf{F}}$ and $\vec{\mathbf{v}}$ are in phase, we conclude that at resonance, the applied force is in phase with the velocity and the power transferred to the oscillator is a maximum.

Figure 15.23 is a graph of amplitude as a function of driving frequency for a forced oscillator with and without damping. Notice that the amplitude increases with decreasing damping ($b \to 0$) and that the resonance curve broadens as the damping increases. In the absence of a damping force ($b = 0$), we see from Equation 15.36 that the steady-state amplitude approaches infinity as ω approaches ω_0. In other words, if there are no losses in the system and we continue to drive an initially motionless oscillator with a periodic force that is in phase with the velocity, the amplitude of motion builds without limit (see the red-brown curve in Fig. 15.23). This limitless building does not occur in practice because some damping is always present in reality.

Later in this book we shall see that resonance appears in other areas of physics. For example, certain electric circuits have natural frequencies and can be set into strong resonance by a varying voltage applied at a given frequency. A bridge has natural frequencies that can be set into resonance by an appropriate driving force. A dramatic example of such resonance occurred in 1940 when the Tacoma Narrows Bridge in the state of Washington was destroyed by resonant vibrations. Although the winds were not particularly strong on that occasion, the "flapping" of the wind across the roadway (think of the "flapping" of a flag in a strong wind) provided a periodic driving force whose frequency matched that of the bridge. The resulting oscillations of the bridge caused it to ultimately collapse (Fig. 15.24) because the bridge design had inadequate built-in safety features.

When the frequency ω of the driving force equals the natural frequency ω_0 of the oscillator, resonance occurs.

$b = 0$
Undamped

Small b

Large b

ω_0 · ω

Figure 15.23 Graph of amplitude versus frequency for a damped oscillator when a periodic driving force is present. Notice that the shape of the resonance curve depends on the size of the damping coefficient b.

Figure 15.24 (a) In 1940, turbulent winds set up torsional vibrations in the Tacoma Narrows Bridge, causing it to oscillate at a frequency near one of the natural frequencies of the bridge structure. (b) Once established, this resonance condition led to the bridge's collapse. (Mathematicians and physicists are currently challenging some aspects of this interpretation.)

AP Photos

© Topham/The Image Works

a

b
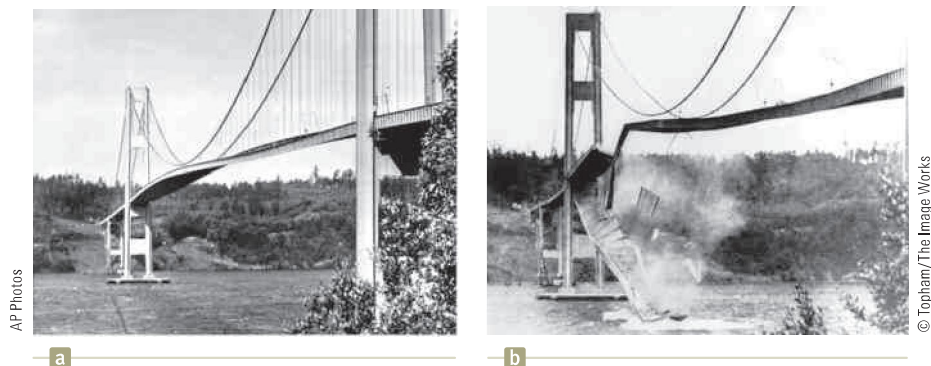
Many other examples of resonant vibrations can be cited. A resonant vibration you may have experienced is the "singing" of telephone wires in the wind. Machines often break if one vibrating part is in resonance with some other moving part. Soldiers marching in cadence across a bridge have been known to set up resonant vibrations in the structure and thereby cause it to collapse. Whenever any real physical system is driven near its resonance frequency, you can expect oscillations of very large amplitudes.

Summary

Concepts and Principles

The kinetic energy and potential energy for an object of mass m oscillating at the end of a spring of force constant k vary with time and are given by

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) \quad \textbf{(15.19)}$$

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2 \cos^2(\omega t + \phi) \quad \textbf{(15.20)}$$

The total energy of a simple harmonic oscillator is a constant of the motion and is given by

$$E = \tfrac{1}{2}kA^2 \quad \textbf{(15.21)}$$

A **simple pendulum** of length L can be modeled to move in simple harmonic motion for small angular displacements from the vertical. Its period is

$$T = 2\pi\sqrt{\frac{L}{g}} \quad \textbf{(15.26)}$$

A **physical pendulum** is an extended object that, for small angular displacements, can be modeled to move in simple harmonic motion about a pivot that does not go through the center of mass. The period of this motion is

$$T = 2\pi\sqrt{\frac{I}{mgd}} \quad \textbf{(15.28)}$$

where I is the moment of inertia of the object about an axis through the pivot and d is the distance from the pivot to the center of mass of the object.

If an oscillator experiences a damping force $\vec{\mathbf{R}} = -b\vec{\mathbf{v}}$, its position for small damping is described by

$$x = Ae^{-(b/2m)t}\cos(\omega t + \phi) \quad \textbf{(15.32)}$$

where

$$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \quad \textbf{(15.33)}$$

If an oscillator is subject to a sinusoidal driving force that is described by $F(t) = F_0 \sin \omega t$, it exhibits **resonance,** in which the amplitude is largest when the driving frequency ω matches the natural frequency $\omega_0 = \sqrt{k/m}$ of the oscillator.

Analysis Model for Problem Solving

Particle in Simple Harmonic Motion If a particle is subject to a force of the form of Hooke's law $F = -kx$, the particle exhibits **simple harmonic motion.** Its position is described by
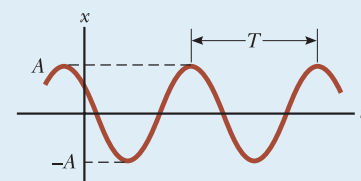
$$x(t) = A\cos(\omega t + \phi) \quad \textbf{(15.6)}$$

where A is the **amplitude** of the motion, ω is the **angular frequency,** and ϕ is the **phase constant.** The value of ϕ depends on the initial position and initial velocity of the particle.

The **period** of the oscillation of the particle is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \quad \textbf{(15.13)}$$

and the inverse of the period is the **frequency.**

Objective Questions

1. If a simple pendulum oscillates with small amplitude and its length is doubled, what happens to the frequency of its motion? (a) It doubles. (b) It becomes $\sqrt{2}$ times as large. (c) It becomes half as large. (d) It becomes $1/\sqrt{2}$ times as large. (e) It remains the same.

2. You attach a block to the bottom end of a spring hanging vertically. You slowly let the block move down and find that it hangs at rest with the spring stretched by 15.0 cm. Next, you lift the block back up to the initial position and release it from rest with the spring unstretched. What maximum distance does it move down? (a) 7.5 cm (b) 15.0 cm (c) 30.0 cm (d) 60.0 cm (e) The distance cannot be determined without knowing the mass and spring constant.

3. A block–spring system vibrating on a frictionless, horizontal surface with an amplitude of 6.0 cm has an energy of 12 J. If the block is replaced by one whose mass is twice the mass of the original block and the amplitude of the motion is again 6.0 cm, what is the energy of the system? (a) 12 J (b) 24 J (c) 6 J (d) 48 J (e) none of those answers

4. An object–spring system moving with simple harmonic motion has an amplitude A. When the kinetic energy of the object equals twice the potential energy stored in the spring, what is the position x of the object? (a) A (b) $\frac{1}{3}A$ (c) $A/\sqrt{3}$ (d) 0 (e) none of those answers

5. An object of mass 0.40 kg, hanging from a spring with a spring constant of 8.0 N/m, is set into an up-and-down simple harmonic motion. What is the magnitude of the acceleration of the object when it is at its maximum displacement of 0.10 m? (a) zero (b) 0.45 m/s² (c) 1.0 m/s² (d) 2.0 m/s² (e) 2.4 m/s²

6. A runaway railroad car, with mass 3.0×10^5 kg, coasts across a level track at 2.0 m/s when it collides elastically with a spring-loaded bumper at the end of the track. If the spring constant of the bumper is 2.0×10^6 N/m, what is the maximum compression of the spring during the collision? (a) 0.77 m (b) 0.58 m (c) 0.34 m (d) 1.07 m (e) 1.24 m

7. The position of an object moving with simple harmonic motion is given by $x = 4 \cos (6\pi t)$, where x is in meters and t is in seconds. What is the period of the oscillating system? (a) 4 s (b) $\frac{1}{6}$ s (c) $\frac{1}{3}$ s (d) 6π s (e) impossible to determine from the information given

8. If an object of mass m attached to a light spring is replaced by one of mass $9m$, the frequency of the vibrating system changes by what factor? (a) $\frac{1}{9}$ (b) $\frac{1}{3}$ (c) 3.0 (d) 9.0 (e) 6.0

9. You stand on the end of a diving board and bounce to set it into oscillation. You find a maximum response in terms of the amplitude of oscillation of the end of the board when you bounce at frequency f. You now move to the middle of the board and repeat the experiment. Is the resonance frequency for forced oscillations at this point (a) higher, (b) lower, or (c) the same as f?

10. A mass–spring system moves with simple harmonic motion along the x axis between turning points at $x_1 = 20$ cm and $x_2 = 60$ cm. For parts (i) through (iii), choose from the same five possibilities. **(i)** At which position does the particle have the greatest magnitude of momentum? (a) 20 cm (b) 30 cm (c) 40 cm (d) some other position (e) The greatest value occurs at multiple points. **(ii)** At which position does the particle have greatest kinetic energy? **(iii)** At which position does the particle-spring system have the greatest total energy?

11. A block with mass $m = 0.1$ kg oscillates with amplitude $A = 0.1$ m at the end of a spring with force constant $k = 10$ N/m on a frictionless, horizontal surface. Rank the periods of the following situations from greatest to smallest. If any periods are equal, show their equality in your ranking. (a) The system is as described above. (b) The system is as described in situation (a) except the amplitude is 0.2 m. (c) The situation is as described in situation (a) except the mass is 0.2 kg. (d) The situation is as described in situation (a) except the spring has force constant 20 N/m. (e) A small resistive force makes the motion underdamped.

12. For a simple harmonic oscillator, answer yes or no to the following questions. (a) Can the quantities position and velocity have the same sign? (b) Can velocity and acceleration have the same sign? (c) Can position and acceleration have the same sign?
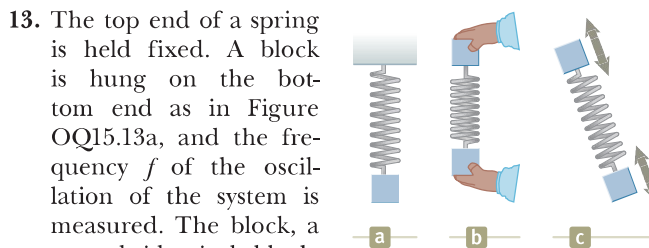
13. The top end of a spring is held fixed. A block is hung on the bottom end as in Figure OQ15.13a, and the frequency f of the oscillation of the system is measured. The block, a second identical block, and the spring are carried up in a space shuttle to Earth orbit. The two blocks are attached to the ends of the spring. The spring is compressed without making adjacent coils touch (Fig. OQ15.13b), and the system is released to oscillate while floating within the shuttle cabin (Fig. OQ15.13c). What is the frequency of oscillation for this system in terms of f? (a) $f/2$ (b) $f/\sqrt{2}$ (c) f (d) $\sqrt{2}f$ (e) $2f$

Figure OQ15.13

14. Which of the following statements is *not* true regarding a mass–spring system that moves with simple harmonic motion in the absence of friction? (a) The total energy of the system remains constant. (b) The energy of the system is continually transformed between kinetic and potential energy. (c) The total energy of the system is proportional to the square of the amplitude. (d) The potential energy stored in the system is greatest when the mass passes through the equilibrium position. (e) The velocity of the oscillating mass has its maximum value when the mass passes through the equilibrium position.

15. A simple pendulum has a period of 2.5 s. **(i)** What is its period if its length is made four times larger? (a) 1.25 s (b) 1.77 s (c) 2.5 s (d) 3.54 s (e) 5 s **(ii)** What is its period if the length is held constant at its initial value and the mass of the suspended bob is made four times larger? Choose from the same possibilities.

16. A simple pendulum is suspended from the ceiling of a stationary elevator, and the period is determined. **(i)** When the elevator accelerates upward, is the period (a) greater, (b) smaller, or (c) unchanged? **(ii)** When the elevator has a downward acceleration, is the period (a) greater, (b) smaller, or (c) unchanged? **(iii)** When the elevator moves with constant upward velocity, is the period of the pendulum (a) greater, (b) smaller, or (c) unchanged?

17. A particle on a spring moves in simple harmonic motion along the x axis between turning points at $x_1 = 100$ cm and $x_2 = 140$ cm. **(i)** At which of the following positions does the particle have maximum speed? (a) 100 cm (b) 110 cm (c) 120 cm (d) at none of those positions **(ii)** At which position does it have maximum acceleration? Choose from the same possibilities as in part (i). **(iii)** At which position is the greatest net force exerted on the particle? Choose from the same possibilities as in part (i).

Conceptual Questions

1. denotes answer available in *Student Solutions Manual/Study Guide*

1. You are looking at a small, leafy tree. You do not notice any breeze, and most of the leaves on the tree are motionless. One leaf, however, is fluttering back and forth wildly. After a while, that leaf stops moving and you notice a different leaf moving much more than all the others. Explain what could cause the large motion of one particular leaf.

2. The equations listed together on page 38 give position as a function of time, velocity as a function of time, and velocity as a function of position for an object moving in a straight line with constant acceleration. The quantity v_{xi} appears in every equation. (a) Do any of these equations apply to an object moving in a straight line with simple harmonic motion? (b) Using a similar format, make a table of equations describing simple harmonic motion. Include equations giving acceleration as a function of time and acceleration as a function of position. State the equations in such a form that they apply equally to a block–spring system, to a pendulum, and to other vibrating systems. (c) What quantity appears in every equation?

3. (a) If the coordinate of a particle varies as $x = -A \cos \omega t$, what is the phase constant in Equation 15.6? (b) At what position is the particle at $t = 0$?

4. A pendulum bob is made from a sphere filled with water. What would happen to the frequency of vibration of this pendulum if there were a hole in the sphere that allowed the water to leak out slowly?

5. Figure CQ15.5 shows graphs of the potential energy of four different systems versus the position of a particle in each system. Each particle is set into motion with a push at an arbitrarily chosen location. Describe its subsequent motion in each case (a), (b), (c), and (d).

6. A student thinks that any real vibration must be damped. Is the student correct? If so, give convincing reasoning. If not, give an example of a real vibration that keeps constant amplitude forever if the system is isolated.

7. The mechanical energy of an undamped block–spring system is constant as kinetic energy transforms to elastic potential energy and vice versa. For comparison, explain what happens to the energy of a damped oscillator in terms of the mechanical, potential, and kinetic energies.

8. Is it possible to have damped oscillations when a system is at resonance? Explain.

9. Will damped oscillations occur for any values of b and k? Explain.

10. If a pendulum clock keeps perfect time at the base of a mountain, will it also keep perfect time when it is moved to the top of the mountain? Explain.

11. Is a bouncing ball an example of simple harmonic motion? Is the daily movement of a student from home to school and back simple harmonic motion? Why or why not?

12. A simple pendulum can be modeled as exhibiting simple harmonic motion when θ is small. Is the motion periodic when θ is large?

13. Consider the simplified single-piston engine in Figure CQ15.13. Assuming the wheel rotates with constant angular speed, explain why the piston rod oscillates in simple harmonic motion.
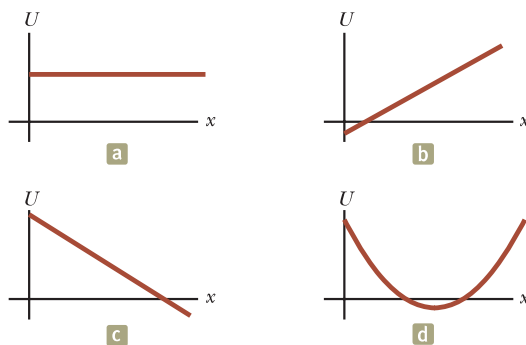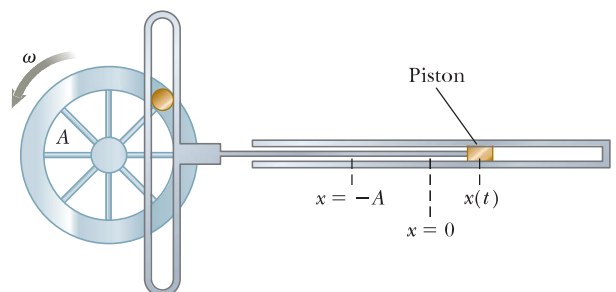

Figure CQ15.5

Figure CQ15.13

Problems

The problems found in this chapter may be assigned online in Enhanced WebAssign

1. straightforward; **2.** intermediate;
3. challenging

1. full solution available in the *Student Solutions Manual/Study Guide*

AMT Analysis Model tutorial available in Enhanced WebAssign

GP Guided Problem

M Master It tutorial available in Enhanced WebAssign

W Watch It video solution available in Enhanced WebAssign

Note: Ignore the mass of every spring, except in Problems 76 and 87.

Section 15.1 Motion of an Object Attached to a Spring

Problems 17, 18, 19, 22, and 59 in Chapter 7 can also be assigned with this section.

1. A 0.60-kg block attached to a spring with force constant 130 N/m is free to move on a frictionless, horizontal surface as in Figure 15.1. The block is released from rest when the spring is stretched 0.13 m. At the instant the block is released, find (a) the force on the block and (b) its acceleration.

2. When a 4.25-kg object is placed on top of a vertical spring, the spring compresses a distance of 2.62 cm. What is the force constant of the spring?

Section 15.2 Analysis Model: Particle in Simple Harmonic Motion

3. A vertical spring stretches 3.9 cm when a 10-g object is hung from it. The object is replaced with a block of mass 25 g that oscillates up and down in simple harmonic motion. Calculate the period of motion.

4. In an engine, a piston oscillates with simple harmonic motion so that its position varies according to the expression

$$x = 5.00 \cos\left(2t + \frac{\pi}{6}\right)$$

where x is in centimeters and t is in seconds. At $t = 0$, find (a) the position of the particle, (b) its velocity, and (c) its acceleration. Find (d) the period and (e) the amplitude of the motion.

5. The position of a particle is given by the expression $x = 4.00 \cos(3.00\pi t + \pi)$, where x is in meters and t is in seconds. Determine (a) the frequency and (b) period of the motion, (c) the amplitude of the motion, (d) the phase constant, and (e) the position of the particle at $t = 0.250$ s.

6. A piston in a gasoline engine is in simple harmonic motion. The engine is running at the rate of 3 600 rev/min. Taking the extremes of its position relative to its center point as ±5.00 cm, find the magnitudes of the (a) maximum velocity and (b) maximum acceleration of the piston.

7. A 1.00-kg object is attached to a horizontal spring. The spring is initially stretched by 0.100 m, and the object is released from rest there. It proceeds to move without friction. The next time the speed of the object is zero is 0.500 s later. What is the maximum speed of the object?

8. A simple harmonic oscillator takes 12.0 s to undergo five complete vibrations. Find (a) the period of its motion, (b) the frequency in hertz, and (c) the angular frequency in radians per second.

9. A 7.00-kg object is hung from the bottom end of a vertical spring fastened to an overhead beam. The object is set into vertical oscillations having a period of 2.60 s. Find the force constant of the spring.

10. At an outdoor market, a bunch of bananas attached to the bottom of a vertical spring of force constant 16.0 N/m is set into oscillatory motion with an amplitude of 20.0 cm. It is observed that the maximum speed of the bunch of bananas is 40.0 cm/s. What is the weight of the bananas in newtons?

11. A vibration sensor, used in testing a washing machine, consists of a cube of aluminum 1.50 cm on edge mounted on one end of a strip of spring steel (like a hacksaw blade) that lies in a vertical plane. The strip's mass is small compared with that of the cube, but the strip's length is large compared with the size of the cube. The other end of the strip is clamped to the frame of the washing machine that is not operating. A horizontal force of 1.43 N applied to the cube is required to hold it 2.75 cm away from its equilibrium position. If it is released, what is its frequency of vibration?

12. (a) A hanging spring stretches by 35.0 cm when an object of mass 450 g is hung on it at rest. In this situation, we define its position as $x = 0$. The object is pulled down an additional 18.0 cm and released from rest to oscillate without friction. What is its position x at a moment 84.4 s later? (b) Find the distance traveled by the vibrating object in part (a). (c) **What If?** Another hanging spring stretches by 35.5 cm when an object of mass 440 g is hung on it at rest. We define this new position as $x = 0$. This object is also pulled down an additional 18.0 cm and released from rest to oscillate without friction. Find its position 84.4 s later. (d) Find the distance traveled by the object in part (c). (e) Why are the answers to parts (a) and (c) so different when the initial data in parts (a) and (c) are so similar and the answers to parts (b) and (d) are relatively close? Does this circumstance reveal a fundamental difficulty in calculating the future?

13. **Review.** A particle moves along the x axis. It is initially at the position 0.270 m, moving with velocity 0.140 m/s and acceleration -0.320 m/s^2. Suppose it moves as a particle under constant acceleration for 4.50 s. Find (a) its position and (b) its velocity at the end of this time interval. Next, assume it moves as a particle in simple harmonic motion for 4.50 s and $x = 0$ is its equilibrium position. Find (c) its position and (d) its velocity at the end of this time interval.

14. A ball dropped from a height of 4.00 m makes an elastic collision with the ground. Assuming no mechanical energy is lost due to air resistance, (a) show that the ensuing motion is periodic and (b) determine the period of the motion. (c) Is the motion simple harmonic? Explain.

15. A particle moving along the x axis in simple harmonic motion starts from its equilibrium position, the origin, at $t = 0$ and moves to the right. The amplitude of its motion is 2.00 cm, and the frequency is 1.50 Hz. (a) Find an expression for the position of the particle as a function of time. Determine (b) the maximum speed of the particle and (c) the earliest time $(t > 0)$ at which the particle has this speed. Find (d) the maximum positive acceleration of the particle and (e) the earliest time $(t > 0)$ at which the particle has this acceleration. (f) Find the total distance traveled by the particle between $t = 0$ and $t = 1.00$ s.

16. The initial position, velocity, and acceleration of an object moving in simple harmonic motion are x_i, v_i, and a_i; the angular frequency of oscillation is ω. (a) Show that the position and velocity of the object for all time can be written as

$$x(t) = x_i \cos \omega t + \left(\frac{v_i}{\omega}\right) \sin \omega t$$

$$v(t) = -x_i \omega \sin \omega t + v_i \cos \omega t$$

(b) Using A to represent the amplitude of the motion, show that

$$v^2 - ax = v_i^2 - a_i x_i = \omega^2 A^2$$

17. A particle moves in simple harmonic motion with a frequency of 3.00 Hz and an amplitude of 5.00 cm. (a) Through what total distance does the particle move during one cycle of its motion? (b) What is its maximum speed? Where does this maximum speed occur? (c) Find the maximum acceleration of the particle. Where in the motion does the maximum acceleration occur?

18. A 1.00-kg glider attached to a spring with a force constant of 25.0 N/m oscillates on a frictionless, horizontal air track. At $t = 0$, the glider is released from rest at $x = -3.00$ cm (that is, the spring is compressed by 3.00 cm). Find (a) the period of the glider's motion, (b) the maximum values of its speed and acceleration, and (c) the position, velocity, and acceleration as functions of time.

19. A 0.500-kg object attached to a spring with a force constant of 8.00 N/m vibrates in simple harmonic motion with an amplitude of 10.0 cm. Calculate the maximum

value of its (a) speed and (b) acceleration, (c) the speed and (d) the acceleration when the object is 6.00 cm from the equilibrium position, and (e) the time interval required for the object to move from $x = 0$ to $x = 8.00$ cm.

20. You attach an object to the bottom end of a hanging vertical spring. It hangs at rest after extending the spring 18.3 cm. You then set the object vibrating. (a) Do you have enough information to find its period? (b) Explain your answer and state whatever you can about its period.

Section 15.3 Energy of the Simple Harmonic Oscillator

21. To test the resiliency of its bumper during low-speed collisions, a 1 000-kg automobile is driven into a brick wall. The car's bumper behaves like a spring with a force constant 5.00×10^6 N/m and compresses 3.16 cm as the car is brought to rest. What was the speed of the car before impact, assuming no mechanical energy is transformed or transferred away during impact with the wall?

22. A 200-g block is attached to a horizontal spring and executes simple harmonic motion with a period of 0.250 s. The total energy of the system is 2.00 J. Find (a) the force constant of the spring and (b) the amplitude of the motion.

23. A block of unknown mass is attached to a spring with a spring constant of 6.50 N/m and undergoes simple harmonic motion with an amplitude of 10.0 cm. When the block is halfway between its equilibrium position and the end point, its speed is measured to be 30.0 cm/s. Calculate (a) the mass of the block, (b) the period of the motion, and (c) the maximum acceleration of the block.

24. A block–spring system oscillates with an amplitude of 3.50 cm. The spring constant is 250 N/m and the mass of the block is 0.500 kg. Determine (a) the mechanical energy of the system, (b) the maximum speed of the block, and (c) the maximum acceleration.

25. A particle executes simple harmonic motion with an amplitude of 3.00 cm. At what position does its speed equal half of its maximum speed?

26. The amplitude of a system moving in simple harmonic motion is doubled. Determine the change in (a) the total energy, (b) the maximum speed, (c) the maximum acceleration, and (d) the period.

27. A 50.0-g object connected to a spring with a force constant of 35.0 N/m oscillates with an amplitude of 4.00 cm on a frictionless, horizontal surface. Find (a) the total energy of the system and (b) the speed of the object when its position is 1.00 cm. Find (c) the kinetic energy and (d) the potential energy when its position is 3.00 cm.

28. A 2.00-kg object is attached to a spring and placed on a frictionless, horizontal surface. A horizontal force of 20.0 N is required to hold the object at rest when it is pulled 0.200 m from its equilibrium position (the origin of the x axis). The object is now released

from rest from this stretched position, and it subsequently undergoes simple harmonic oscillations. Find (a) the force constant of the spring, (b) the frequency of the oscillations, and (c) the maximum speed of the object. (d) Where does this maximum speed occur? (e) Find the maximum acceleration of the object. (f) Where does the maximum acceleration occur? (g) Find the total energy of the oscillating system. Find (h) the speed and (i) the acceleration of the object when its position is equal to one-third the maximum value.

29. A simple harmonic oscillator of amplitude A has a total energy E. Determine (a) the kinetic energy and (b) the potential energy when the position is one-third the amplitude. (c) For what values of the position does the kinetic energy equal one-half the potential energy? (d) Are there any values of the position where the kinetic energy is greater than the maximum potential energy? Explain.

30. Review. A 65.0-kg bungee jumper steps off a bridge **GP** with a light bungee cord tied to her body and to the bridge. The unstretched length of the cord is 11.0 m. The jumper reaches the bottom of her motion 36.0 m below the bridge before bouncing back. We wish to find the time interval between her leaving the bridge and her arriving at the bottom of her motion. Her overall motion can be separated into an 11.0-m free fall and a 25.0-m section of simple harmonic oscillation. (a) For the free-fall part, what is the appropriate analysis model to describe her motion? (b) For what time interval is she in free fall? (c) For the simple harmonic oscillation part of the plunge, is the system of the bungee jumper, the spring, and the Earth isolated or non-isolated? (d) From your response in part (c) find the spring constant of the bungee cord. (e) What is the location of the equilibrium point where the spring force balances the gravitational force exerted on the jumper? (f) What is the angular frequency of the oscillation? (g) What time interval is required for the cord to stretch by 25.0 m? (h) What is the total time interval for the entire 36.0-m drop?

31. Review. A 0.250-kg block resting on a frictionless, horizontal surface is attached to a spring whose force constant is 83.8 N/m as in Figure P15.31. A horizontal force $\vec{F}$ causes the spring to stretch a distance of 5.46 cm from its equilibrium position. (a) Find the magnitude of $\vec{F}$. (b) What is the total energy stored in the system when the spring is stretched? (c) Find the magnitude of the acceleration of the block just after the applied force is removed. (d) Find the speed of the block when it first reaches the equilibrium position. (e) If the surface is not frictionless but the block still reaches the equilibrium position, would your answer to part (d) be larger or smaller? (f) What other information would you need to know to find the actual answer to part (d) in this case? (g) What is the largest value of the coefficient of
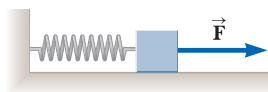

Figure P15.31

friction that would allow the block to reach the equilibrium position?

32. A 326-g object is attached to a spring and executes sim-**AMT** ple harmonic motion with a period of 0.250 s. If the total energy of the system is 5.83 J, find (a) the maximum speed of the object, (b) the force constant of the spring, and (c) the amplitude of the motion.

Section 15.4 Comparing Simple Harmonic Motion with Uniform Circular Motion

33. While driving behind a car traveling at 3.00 m/s, you notice that one of the car's tires has a small hemispherical bump on its rim as shown in Figure P15.33. (a) Explain why the bump, from your viewpoint behind the car, executes simple harmonic motion. (b) If the radii of the car's tires are 0.300 m, what is the bump's period of oscillation?
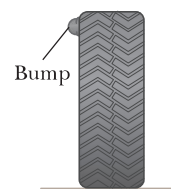

Bump

Figure P15.33

Section 15.5 The Pendulum

Problem 68 in Chapter 1 can also be assigned with this section.

34. A "seconds pendulum" is one that moves through its equilibrium position once each second. (The period of the pendulum is precisely 2 s.) The length of a seconds pendulum is 0.992 7 m at Tokyo, Japan, and 0.994 2 m at Cambridge, England. What is the ratio of the free-fall accelerations at these two locations?

35. A simple pendulum makes 120 complete oscillations in 3.00 min at a location where $g = 9.80$ m/s^2. Find (a) the period of the pendulum and (b) its length.

36. A particle of mass m slides without friction inside a hemispherical bowl of radius R. Show that if the particle starts from rest with a small displacement from equilibrium, it moves in simple harmonic motion with an angular frequency equal to that of a simple pendulum of length R. That is, $\omega = \sqrt{g/R}$.

37. A physical pendulum in the form of a planar object **M** moves in simple harmonic motion with a frequency of 0.450 Hz. The pendulum has a mass of 2.20 kg, and the pivot is located 0.350 m from the center of mass. Determine the moment of inertia of the pendulum about the pivot point.

38. A physical pendulum in the form of a planar object moves in simple harmonic motion with a frequency f. The pendulum has a mass m, and the pivot is located a distance d from the center of mass. Determine the moment of inertia of the pendulum about the pivot point.

39. The angular position of a pendulum is represented by the equation $\theta = 0.032\ 0 \cos \omega t$, where θ is in radians and $\omega = 4.43$ rad/s. Determine the period and length of the pendulum.

40. Consider the physical pendulum of Figure 15.17. (a) Represent its moment of inertia about an axis passing

through its center of mass and parallel to the axis passing through its pivot point as I_{CM}. Show that its period is

$$T = 2\pi\sqrt{\frac{I_{CM} + md^2}{mgd}}$$

where d is the distance between the pivot point and the center of mass. (b) Show that the period has a minimum value when d satisfies $md^2 = I_{CM}$.

41. A simple pendulum has a mass of 0.250 kg and a length of 1.00 m. It is displaced through an angle of 15.0° and then released. Using the analysis model of a particle in simple harmonic motion, what are (a) the maximum speed of the bob, (b) its maximum angular acceleration, and (c) the maximum restoring force on the bob? (d) **What If?** Solve parts (a) through (c) again by using analysis models introduced in earlier chapters. (e) Compare the answers.

42. A very light rigid rod of length 0.500 m extends straight out from one end of a meterstick. The combination is suspended from a pivot at the upper end of the rod as shown in Figure P15.42. The combination is then pulled out by a small angle and released. (a) Determine the period of oscillation of the system. (b) By what percentage does the period differ from the period of a simple pendulum 1.00 m long?
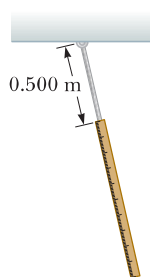
Figure P15.42

43. Review. A simple pendulum is 5.00 m long. What is the period of small oscillations for this pendulum if it is located in an elevator (a) accelerating upward at 5.00 m/s²? (b) Accelerating downward at 5.00 m/s²? (c) What is the period of this pendulum if it is placed in a truck that is accelerating horizontally at 5.00 m/s²?

44. A small object is attached to the end of a string to form a simple pendulum. The period of its harmonic motion is measured for small angular displacements and three lengths. For lengths of 1.000 m, 0.750 m, and 0.500 m, total time intervals for 50 oscillations of 99.8 s, 86.6 s, and 71.1 s are measured with a stopwatch. (a) Determine the period of motion for each length. (b) Determine the mean value of g obtained from these three independent measurements and compare it with the accepted value. (c) Plot T^2 versus L and obtain a value for g from the slope of your best-fit straight-line graph. (d) Compare the value found in part (c) with that obtained in part (b).

45. A watch balance wheel (Fig. P15.45) has a period of oscillation of 0.250 s. The wheel is constructed so that its mass of 20.0 g is concentrated around a rim of radius 0.500 cm. What are (a) the wheel's moment of inertia and (b) the torsion constant of the attached spring?
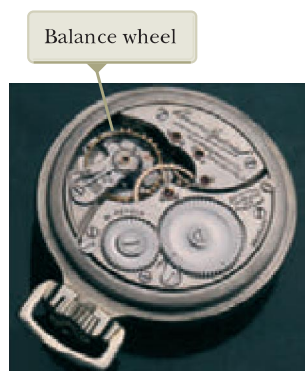
Balance wheel

Figure P15.45

Section 15.6 Damped Oscillations

46. A pendulum with a length of 1.00 m is released from an initial angle of 15.0°. After 1 000 s, its amplitude has been reduced by friction to 5.50°. What is the value of $b/2m$?

47. A 10.6-kg object oscillates at the end of a vertical spring that has a spring constant of 2.05×10^4 N/m. The effect of air resistance is represented by the damping coefficient $b = 3.00$ N · s/m. (a) Calculate the frequency of the damped oscillation. (b) By what percentage does the amplitude of the oscillation decrease in each cycle? (c) Find the time interval that elapses while the energy of the system drops to 5.00% of its initial value.

48. Show that the time rate of change of mechanical energy for a damped, undriven oscillator is given by $dE/dt = -bv^2$ and hence is always negative. To do so, differentiate the expression for the mechanical energy of an oscillator, $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$, and use Equation 15.31.

49. Show that Equation 15.32 is a solution of Equation 15.31 provided that $b^2 < 4mk$.

Section 15.7 Forced Oscillations

50. A baby bounces up and down in her crib. Her mass is 12.5 kg, and the crib mattress can be modeled as a light spring with force constant 700 N/m. (a) The baby soon learns to bounce with maximum amplitude and minimum effort by bending her knees at what frequency? (b) If she were to use the mattress as a trampoline—losing contact with it for part of each cycle—what minimum amplitude of oscillation does she require?

51. As you enter a fine restaurant, you realize that you have accidentally brought a small electronic timer from home instead of your cell phone. In frustration, you drop the timer into a side pocket of your suit coat, not realizing that the timer is operating. The arm of your chair presses the light cloth of your coat against your body at one spot. Fabric with a length L hangs freely below that spot, with the timer at the bottom. At one point during your dinner, the timer goes off and a buzzer and a vibrator turn on and off with a frequency of 1.50 Hz. It makes the hanging part of your coat swing back and forth with remarkably large amplitude, drawing everyone's attention. Find the value of L.

52. A block weighing 40.0 N is suspended from a spring that has a force constant of 200 N/m. The system is undamped ($b = 0$) and is subjected to a harmonic driving force of frequency 10.0 Hz, resulting in a forced-motion amplitude of 2.00 cm. Determine the maximum value of the driving force.

53. A 2.00-kg object attached to a spring moves without friction ($b = 0$) and is driven by an external force given by the expression $F = 3.00 \sin(2\pi t)$, where F is in newtons and t is in seconds. The force constant of the spring is 20.0 N/m. Find (a) the resonance angular frequency of the system, (b) the angular frequency of the driven system, and (c) the amplitude of the motion.

54. Considering an undamped, forced oscillator ($b = 0$), show that Equation 15.35 is a solution of Equation 15.34, with an amplitude given by Equation 15.36.

55. Damping is negligible for a 0.150-kg object hanging from a light, 6.30-N/m spring. A sinusoidal force with an amplitude of 1.70 N drives the system. At what frequency will the force make the object vibrate with an amplitude of 0.440 m?

Additional Problems

56. The mass of the deuterium molecule (D_2) is twice that of the hydrogen molecule (H_2). If the vibrational frequency of H_2 is 1.30×10^{14} Hz, what is the vibrational frequency of D_2? Assume the "spring constant" of attracting forces is the same for the two molecules.

57. An object of mass m moves in simple harmonic motion with amplitude 12.0 cm on a light spring. Its maximum acceleration is 108 cm/s^2. Regard m as a variable. (a) Find the period T of the object. (b) Find its frequency f. (c) Find the maximum speed v_{max} of the object. (d) Find the total energy E of the object–spring system. (e) Find the force constant k of the spring. (f) Describe the pattern of dependence of each of the quantities T, f, v_{max}, E, and k on m.

58. Review. This problem extends the reasoning of Problem 75 in Chapter 9. Two gliders are set in motion on an air track. Glider 1 has mass $m_1 = 0.240$ kg and moves to the right with speed 0.740 m/s. It will have a rear-end collision with glider 2, of mass $m_2 = 0.360$ kg, which initially moves to the right with speed 0.120 m/s. A light spring of force constant 45.0 N/m is attached to the back end of glider 2 as shown in Figure P9.75. When glider 1 touches the spring, superglue instantly and permanently makes it stick to its end of the spring. (a) Find the common speed the two gliders have when the spring is at maximum compression. (b) Find the maximum spring compression distance. The motion after the gliders become attached consists of a combination of (1) the constant-velocity motion of the center of mass of the two-glider system found in part (a) and (2) simple harmonic motion of the gliders relative to the center of mass. (c) Find the energy of the center-of-mass motion. (d) Find the energy of the oscillation.

59. A small ball of mass M is attached to the end of a uniform rod of equal mass M and length L that is pivoted at the top (Fig. P15.59). Determine the tensions in the rod (a) at the pivot and (b) at the point P when the system is stationary. (c) Calculate the period of oscillation for small displacements from equilibrium and (d) determine this period for $L = 2.00$ m.
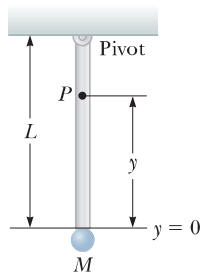

Figure P15.59

60. Review. A rock rests on a concrete sidewalk. An earthquake strikes, making the ground move vertically in simple harmonic motion with a constant frequency of 2.40 Hz and with gradually increasing amplitude. (a) With what amplitude does the ground vibrate when the rock begins to lose contact with the sidewalk? Another rock is sitting on the concrete bottom of a swimming pool full of water. The earthquake produces only vertical motion, so the water does not slosh from side to side. (b) Present a convincing argument that when the ground vibrates with the amplitude found in part (a), the submerged rock also barely loses contact with the floor of the swimming pool.

61. Four people, each with a mass of 72.4 kg, are in a car with a mass of 1 130 kg. An earthquake strikes. The vertical oscillations of the ground surface make the car bounce up and down on its suspension springs, but the driver manages to pull off the road and stop. When the frequency of the shaking is 1.80 Hz, the car exhibits a maximum amplitude of vibration. The earthquake ends, and the four people leave the car as fast as they can. By what distance does the car's undamaged suspension lift the car's body as the people get out?

62. To account for the walking speed of a bipedal or quadrupedal animal, model a leg that is not contacting the ground as a uniform rod of length ℓ, swinging as a physical pendulum through one half of a cycle, in resonance. Let θ_{max} represent its amplitude. (a) Show that the animal's speed is given by the expression

$$v = \frac{\sqrt{6g\ell}\sin\theta_{max}}{\pi}$$

if θ_{max} is sufficiently small that the motion is nearly simple harmonic. An empirical relationship that is based on the same model and applies over a wider range of angles is

$$v = \frac{\sqrt{6g\ell\cos(\theta_{max}/2)}\sin\theta_{max}}{\pi}$$

(b) Evaluate the walking speed of a human with leg length 0.850 m and leg-swing amplitude 28.0°. (c) What leg length would give twice the speed for the same angular amplitude?

63. The free-fall acceleration on Mars is 3.7 m/s^2. (a) What length of pendulum has a period of 1.0 s on Earth? (b) What length of pendulum would have a 1.0-s period on Mars? An object is suspended from a spring with force constant 10 N/m. Find the mass suspended from this spring that would result in a period of 1.0 s (c) on Earth and (d) on Mars.

64. An object attached to a spring vibrates with simple harmonic motion as described by Figure P15.64. For this motion, find (a) the amplitude, (b) the period, (c) the
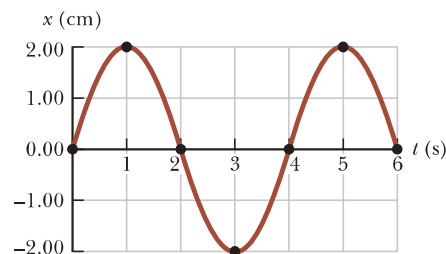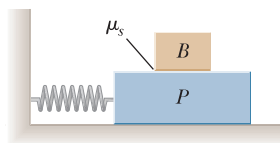

Figure P15.64

angular frequency, (d) the maximum speed, (e) the maximum acceleration, and (f) an equation for its position x as a function of time.

65. **Review.** A large block P attached to a light spring executes horizontal, simple harmonic motion as it slides across a frictionless surface with a frequency $f = 1.50$ Hz. Block B rests on it as shown in Figure P15.65, and the coefficient of static friction between the two is $\mu_s = 0.600$. What maximum amplitude of oscillation can the system have if block B is not to slip?

Figure P15.65
Problems 65 and 66.

66. **Review.** A large block P attached to a light spring executes horizontal, simple harmonic motion as it slides across a frictionless surface with a frequency f. Block B rests on it as shown in Figure P15.65, and the coefficient of static friction between the two is μ_s. What maximum amplitude of oscillation can the system have if block B is not to slip?

67. A pendulum of length L and mass M has a spring of force constant k connected to it at a distance h below its point of suspension (Fig. P15.67). Find the frequency of vibration of the system for small values of the amplitude (small θ). Assume the vertical suspension rod of length L is rigid, but ignore its mass.
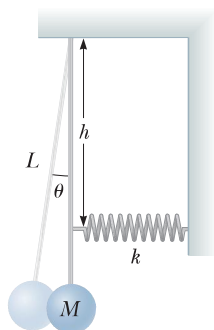
Figure P15.67

68. A block of mass m is connected to two springs of force constants k_1 and k_2 in two ways as shown in Figure P15.68. In both cases, the block moves on a frictionless table after it is displaced from equilibrium and released. Show that in the two cases the block exhibits simple harmonic motion with periods

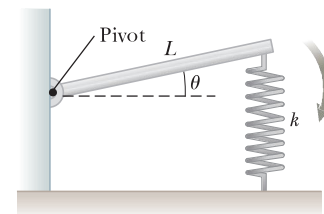(a) $T = 2\pi\sqrt{\dfrac{m(k_1 + k_2)}{k_1 k_2}}$ and (b) $T = 2\pi\sqrt{\dfrac{m}{k_1 + k_2}}$
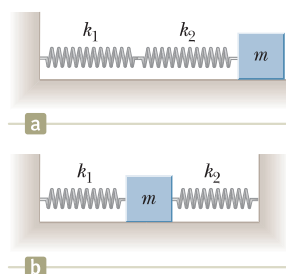
Figure P15.68

69. A horizontal plank of mass 5.00 kg and length 2.00 m is pivoted at one end. The plank's other end is supported by a spring of force constant 100 N/m (Fig. P15.69). The plank is displaced by a small angle θ from its horizontal equilibrium position and released. Find the

angular frequency with which the plank moves with simple harmonic motion.

Figure P15.69
Problems 69 and 70.

**70.** A horizontal plank of mass m and length L is pivoted at one end. The plank's other end is supported by a spring of force constant k (Fig. P15.69). The plank is displaced by a small angle θ from its horizontal equilibrium position and released. Find the angular frequency with which the plank moves with simple harmonic motion.

71. **Review.** A particle of mass 4.00 kg is attached to a spring with a force constant of 100 N/m. It is oscillating on a frictionless, horizontal surface with an amplitude of 2.00 m. A 6.00-kg object is dropped vertically on top of the 4.00-kg object as it passes through its equilibrium point. The two objects stick together. (a) What is the new amplitude of the vibrating system after the collision? (b) By what factor has the period of the system changed? (c) By how much does the energy of the system change as a result of the collision? (d) Account for the change in energy.

72. A ball of mass m is connected to two rubber bands of length L, each under tension T as shown in Figure P15.72. The ball is displaced by a small distance y perpendicular to the length of the rubber bands. Assuming the tension does not change, show that (a) the restoring force is $-(2T/L)y$ and (b) the system exhibits simple harmonic motion with an angular frequency $\omega = \sqrt{2T/mL}$.
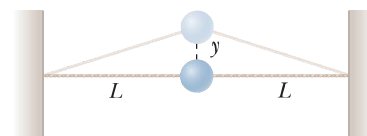
Figure P15.72

73. **Review.** One end of a light spring with force constant $k = 100$ N/m is attached to a vertical wall. A light string is tied to the other end of the horizontal spring. As shown in Figure P15.73, the string changes from horizontal to vertical as it passes over a pulley of mass M in the shape of a solid disk of radius $R = 2.00$ cm. The pulley is free to turn on a fixed, smooth axle. The vertical section of the string supports an object of mass $m = 200$ g. The string does not slip at its contact with the pulley. The object is pulled downward a small distance and released. (a) What is the angular frequency ω of oscillation of the object in terms of the mass M? (b) What is the highest possible value of the angular frequency of oscillation of
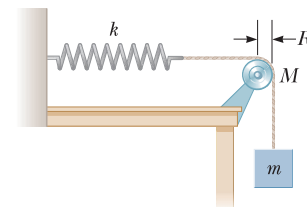
Figure P15.73

the object? (c) What is the highest possible value of the angular frequency of oscillation of the object if the pulley radius is doubled to $R = 4.00$ cm?

74. People who ride motorcycles and bicycles learn to look out for bumps in the road and especially for *washboarding*, a condition in which many equally spaced ridges are worn into the road. What is so bad about washboarding? A motorcycle has several springs and shock absorbers in its suspension, but you can model it as a single spring supporting a block. You can estimate the force constant by thinking about how far the spring compresses when a heavy rider sits on the seat. A motorcyclist traveling at highway speed must be particularly careful of washboard bumps that are a certain distance apart. What is the order of magnitude of their separation distance?

75. A simple pendulum with a length of 2.23 m and a mass of 6.74 kg is given an initial speed of 2.06 m/s at its equilibrium position. Assume it undergoes simple harmonic motion. Determine (a) its period, (b) its total energy, and (c) its maximum angular displacement.

76. When a block of mass M, connected to the end of a spring of mass $m_s = 7.40$ g and force constant k, is set into simple harmonic motion, the period of its motion is

$$T = 2\pi\sqrt{\frac{M + (m_s/3)}{k}}$$

A two-part experiment is conducted with the use of blocks of various masses suspended vertically from the spring as shown in Figure P15.76. (a) Static extensions of 17.0, 29.3, 35.3, 41.3, 47.1, and 49.3 cm are measured for M values of 20.0, 40.0, 50.0, 60.0, 70.0, and 80.0 g, respectively. Construct a graph of Mg versus x and perform a linear least-squares fit to the data. (b) From the slope of your graph, determine a value for k for this spring. (c) The system is now set into simple harmonic motion, and periods are measured with a stopwatch. With $M = 80.0$ g, the total time interval required for ten oscillations is measured to be 13.41 s. The experiment is repeated with M values of 70.0, 60.0, 50.0, 40.0, and 20.0 g, with corresponding time intervals for ten oscillations of 12.52, 11.67, 10.67, 9.62, and 7.03 s. Make a table of these masses and times. (d) Compute the experimental value for T from each of these measurements. (e) Plot a graph of T^2 versus M and (f) determine a value for k from the slope of the linear least-squares fit through the data points. (g) Compare this value of k with that obtained in part (b). (h) Obtain a value for m_s from your graph and compare it with the given value of 7.40 g.

Figure P15.76

77. **Review.** A light balloon filled with helium of density 0.179 kg/m³ is tied to a light string of length $L = 3.00$ m. The string is tied to the ground forming an "inverted" simple pendulum (Fig. 15.77a). If the balloon is displaced slightly from equilibrium as in Figure P15.77b and released, (a) show that the motion is simple harmonic and (b) determine the period of

the motion. Take the density of air to be 1.20 kg/m³. *Hint:* Use an analogy with the simple pendulum and see Chapter 14. Assume the air applies a buoyant force on the balloon but does not otherwise affect its motion.
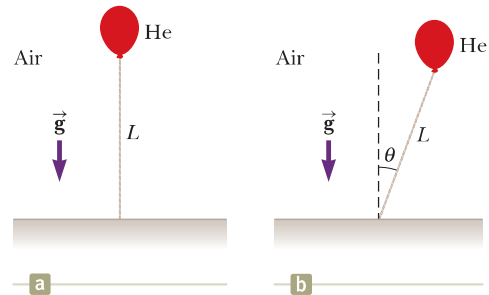
Figure P15.77

78. Consider the damped oscillator illustrated in Figure 15.20. The mass of the object is 375 g, the spring constant is 100 N/m, and $b = 0.100$ N · s/m. (a) Over what time interval does the amplitude drop to half its initial value? (b) **What If?** Over what time interval does the mechanical energy drop to half its initial value? (c) Show that, in general, the fractional rate at which the amplitude decreases in a damped harmonic oscillator is one-half the fractional rate at which the mechanical energy decreases.

79. A particle with a mass of 0.500 kg is attached to a horizontal spring with a force constant of 50.0 N/m. At the moment $t = 0$, the particle has its maximum speed of 20.0 m/s and is moving to the left. (a) Determine the particle's equation of motion, specifying its position as a function of time. (b) Where in the motion is the potential energy three times the kinetic energy? (c) Find the minimum time interval required for the particle to move from $x = 0$ to $x = 1.00$ m. (d) Find the length of a simple pendulum with the same period.

80. Your thumb squeaks on a plate you have just washed. Your sneakers squeak on the gym floor. Car tires squeal when you start or stop abruptly. You can make a goblet sing by wiping your moistened finger around its rim. When chalk squeaks on a blackboard, you can see that it makes a row of regularly spaced dashes. As these examples suggest, vibration commonly results when friction acts on a moving elastic object. The oscillation is not simple harmonic motion, but is called *stick-and-slip*. This problem models stick-and-slip motion.

A block of mass m is attached to a fixed support by a horizontal spring with force constant k and negligible mass (Fig. P15.80). Hooke's law describes the spring
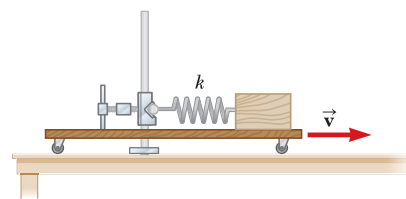
Figure P15.80

both in extension and in compression. The block sits on a long horizontal board, with which it has coefficient of static friction μ_s and a smaller coefficient of kinetic friction μ_k. The board moves to the right at constant speed v. Assume the block spends most of its time sticking to the board and moving to the right with it, so the speed v is small in comparison to the average speed the block has as it slips back toward the left. (a) Show that the maximum extension of the spring from its unstressed position is very nearly given by $\mu_s mg/k$. (b) Show that the block oscillates around an equilibrium position at which the spring is stretched by $\mu_k mg/k$. (c) Graph the block's position versus time. (d) Show that the amplitude of the block's motion is

$$A = \frac{(\mu_s - \mu_k)mg}{k}$$

(e) Show that the period of the block's motion is

$$T = \frac{2(\mu_s - \mu_k)mg}{vk} + \pi\sqrt{\frac{m}{k}}$$

It is the excess of static over kinetic friction that is important for the vibration. "The squeaky wheel gets the grease" because even a viscous fluid cannot exert a force of static friction.

81. **Review.** A lobsterman's buoy is a solid wooden cylinder of radius r and mass M. It is weighted at one end so that it floats upright in calm seawater, having density ρ. A passing shark tugs on the slack rope mooring the buoy to a lobster trap, pulling the buoy down a distance x from its equilibrium position and releasing it. (a) Show that the buoy will execute simple harmonic motion if the resistive effects of the water are ignored. (b) Determine the period of the oscillations.

82. *Why is the following situation impossible?* Your job involves building very small damped oscillators. One of your designs involves a spring–object oscillator with a spring of force constant $k = 10.0$ N/m and an object of mass $m = 1.00$ g. Your design objective is that the oscillator undergo many oscillations as its amplitude falls to 25.0% of its initial value in a certain time interval. Measurements on your latest design show that the amplitude falls to the 25.0% value in 23.1 ms. This time interval is too long for what is needed in your project. To shorten the time interval, you double the damping constant b for the oscillator. This doubling allows you to reach your design objective.

83. Two identical steel balls, each of mass 67.4 g, are moving in opposite directions at 5.00 m/s. They collide head-on and bounce apart elastically. By squeezing one of the balls in a vise while precise measurements are made of the resulting amount of compression, you find that Hooke's law is a good model of the ball's elastic behavior. A force of 16.0 kN exerted by each jaw of the vise reduces the diameter by 0.200 mm. Model the motion of each ball, while the balls are in contact, as one-half of a cycle of simple harmonic motion. Compute the time interval for which the balls are in contact. (If you solved Problem 57 in Chapter 7, compare your results from this problem with your results from that one.)

Challenge Problems

84. A smaller disk of radius r and mass m is attached rigidly to the face of a second larger disk of radius R and mass M as shown in Figure P15.84. The center of the small disk is located at the edge of the large disk. The large disk is mounted at its center on a frictionless axle. The assembly is rotated through a small angle θ from its equilibrium position and released. (a) Show that the speed of the center of the small disk as it passes through the equilibrium position is
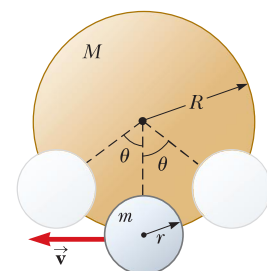

Figure P15.84

$$v = 2\left[\frac{Rg(1 - \cos\theta)}{(M/m) + (r/R)^2 + 2}\right]^{1/2}$$

(b) Show that the period of the motion is

$$T = 2\pi\left[\frac{(M + 2m)R^2 + mr^2}{2mgR}\right]^{1/2}$$

85. An object of mass $m_1 = 9.00$ kg is in equilibrium when connected to a light spring of constant $k = 100$ N/m that is fastened to a wall as shown in Figure P15.85a. A second object, $m_2 = 7.00$ kg, is slowly pushed up against m_1, compressing the spring by the amount $A = 0.200$ m (see Fig. P15.85b). The system is then released, and both objects start moving to the right on the frictionless surface. (a) When m_1 reaches the equilibrium point, m_2 loses contact with m_1 (see Fig. P15.85c) and moves to the right with speed v. Determine the value of v. (b) How far apart are the objects when the spring is fully stretched for the first time (the distance D in Fig. P15.85d)?
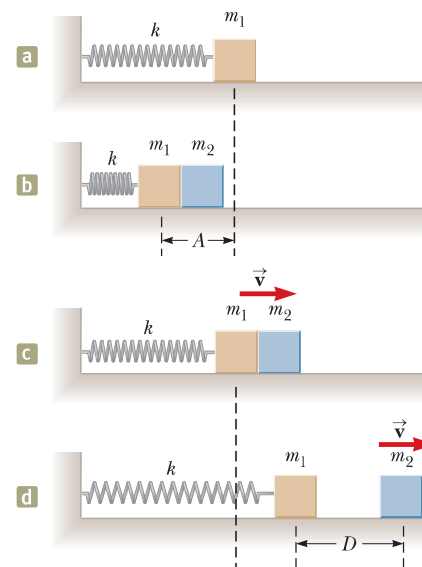

Figure P15.85

86. **Review.** *Why is the following situation impossible?* You are in the high-speed package delivery business. Your competitor in the next building gains the right-of-way to

build an evacuated tunnel just above the ground all the way around the Earth. By firing packages into this tunnel at just the right speed, your competitor is able to send the packages into orbit around the Earth in this tunnel so that they arrive on the exact opposite side of the Earth in a very short time interval. You come up with a competing idea. Figuring that the distance *through* the Earth is shorter than the distance *around* the Earth, you obtain permits to build an evacuated tunnel through the center of the Earth (Fig. P15.86). By simply dropping packages into this tunnel, they fall downward and arrive at the other end of your tunnel, which is in a building right next to the other end of your competitor's tunnel. Because your packages arrive on the other side of the Earth in a shorter time interval, you win the competition and your business flourishes. *Note:* An object at a distance r from the center of the Earth is pulled toward the center of the Earth only by the mass within the sphere of radius r (the reddish region in Fig. P15.86). Assume the Earth has uniform density.
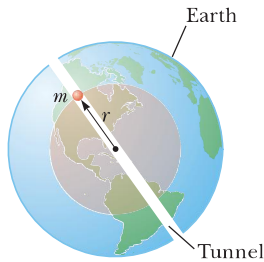

Figure P15.86

87. A block of mass M is connected to a spring of mass m and oscillates in simple harmonic motion on a frictionless, horizontal track (Fig. P15.87). The force constant of the spring is k, and the equilibrium length is ℓ. Assume all portions of the spring oscillate in phase and the velocity of a segment of the spring of length dx
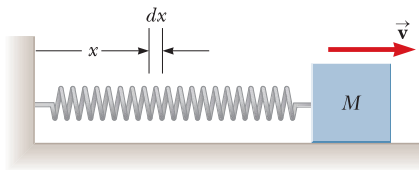

Figure P15.87

is proportional to the distance x from the fixed end; that is, $v_x = (x/\ell)v$. Also, notice that the mass of a segment of the spring is $dm = (m/\ell)\,dx$. Find (a) the kinetic energy of the system when the block has a speed v and (b) the period of oscillation.

88. **Review.** A system consists of a spring with force constant $k = 1\,250$ N/m, length $L = 1.50$ m, and an object of mass $m = 5.00$ kg attached to the end (Fig. P15.88). The object is placed at the level of the point of attachment with the spring unstretched, at position $y_i = L$, and then it is released so that it swings like a pendulum. (a) Find the y position of the object at the lowest point. (b) Will the pendulum's period be greater or less than the period of a simple pendulum with the same mass m and length L? Explain.
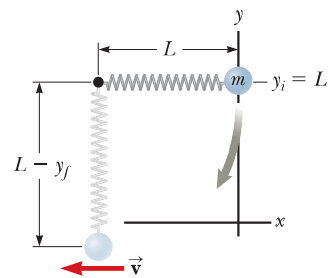

Figure P15.88

89. A light, cubical container of volume a^3 is initially filled with a liquid of mass density ρ as shown in Figure P15.89a. The cube is initially supported by a light string to form a simple pendulum of length L_i, measured from the center of mass of the filled container, where $L_i \gg a$. The liquid is allowed to flow from the bottom of the container at a constant rate (dM/dt). At any time t, the level of the liquid in the container is h and the length of the pendulum is L (measured relative to the instantaneous center of mass) as shown in Figure P15.89b. (a) Find the period of the pendulum as a function of time. (b) What is the period of the pendulum after the liquid completely runs out of the container?
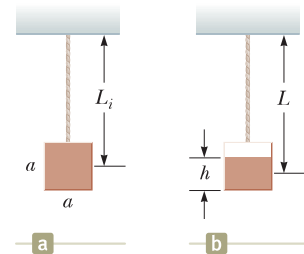

Figure P15.89